Digital Signal Processing

http://kom.aau.dk/~zt/cources/Digital_signal_processing/

Solutions of Lecture 5 (MM5)

Exercise 5.1.

Compute the DFT of each of the following finite-length sequences considered to be o length N (where N is even).

(a) $x[n] = \delta[n]$

(b) $x[n] = \delta[n - n_0], \qquad 0 \leq n_0 \leq N-1$

(c) $x[n] = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$

Solution:

(a)

$$\begin{aligned} x[n] &= \delta[n] \\ X[k] &= \sum_{n=0}^{N-1} \delta[n] W_N^{kn}, \quad 0 \leq k \leq (N-1) \\ &= 1 \end{aligned}$$

(b)

$$\begin{aligned} x[n] &= \delta[n - n_0], \quad 0 \leq n_0 \leq (N-1) \\ X[k] &= \sum_{n=0}^{N-1} \delta[n - n_0] W_N^{kn}, \quad 0 \leq k \leq (N-1) \\ &= W_N^{kn_0} \end{aligned}$$

(c)

$$\begin{aligned} x[n] &= \begin{cases} a^n, & 0 \leq n \leq (N-1) \\ 0, & \text{otherwise} \end{cases} \\ X[k] &= \sum_{n=0}^{N-1} a^n W_N^{kn}, \quad 0 \leq k \leq (N-1) \\ &= \frac{1 - a^N e^{-j2\pi k}}{1 - a e^{-j(2\pi k)/N}} \\ X[k] &= \frac{1 - a^N}{1 - a e^{-j(2\pi k)/N}} \end{aligned}$$

Exercise 5.2.

Suppose we have two 4-point sequences $x[n]$ and $h[n]$ as follows:

$$x[n] = \cos\left(\frac{\pi n}{2}\right), \qquad n = 0, 1, 2, 3,$$

$$h[n] = 2^n, \qquad n = 0, 1, 2, 3.$$

(a) Calculate the 4-point DFT $X[k]$.

(b) Calculate the 4-point DFT $H[k]$.

(c) Calculate $y[n] = x[n] ④ h[n]$ by doing the circular convolution directly.

(d) Calculate $y[n]$ of part (c) by multiplying the DFTs of $x[n]$ and $h[n]$ and performing an inverse DFT.

Solutions:

(a)

$$x[n] = \cos(\frac{\pi n}{2}), \quad 0 \leq n \leq 3$$

transforms to

$$X[k] = \sum_{n=0}^{3} \cos(\frac{\pi n}{2}) W_4^{kn}, \quad 0 \leq k \leq 3$$

The cosine term contributes only two non-zero values to the summation, giving:

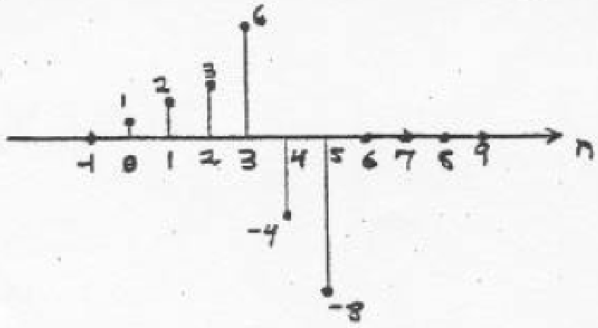
$$\begin{aligned} X[k] &= 1 - e^{-j\pi k}, \quad 0 \leq k \leq 3 \\ &= 1 - W_4^{2k} \end{aligned}$$

(b)

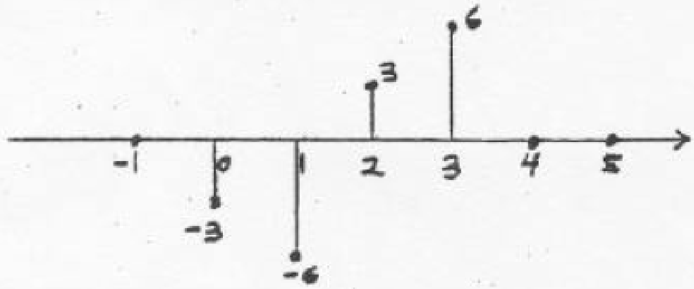
$$h[n] = 2^n, \quad 0 \leq n \leq 3$$

$$
\begin{aligned}
H[k] &= \sum_{n=0}^{3} 2^n W_4^{kn}, \quad 0 \le k \le 3 \\
&= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k}
\end{aligned}
$$

(c) Remember, circular convolution equals linear convolution plus aliasing. We need $N \geq 3 + 4 - 1 = 6$ to avoid aliasing. Since $N = 4$, we expect to get aliasing here. First, find $y[n] = x[n] * h[n]$:



For this problem, aliasing means the last three points ($n = 4, 5, 6$) will wrap-around on top of the first three points, giving $y[n] = x[n] ④ h[n]$:



(d) Using the DFT values we calculated in parts (a) and (b):

$$
\begin{aligned}
Y[k] &= X[k]H[k] \\
&= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k} - W_4^{2k} - 2W_4^{3k} - 4W_4^{4k} - 8W_4^{5k}
\end{aligned}
$$

Since $W_4^{4k} = W_4^{0k}$ and $W_4^{5k} = W_4^k$

$$
Y[k] = -3 - 6W_4^k + 3W_4^{2k} + 6W_4^{3k}, \quad 0 \le k \le 3
$$

Taking the inverse DFT:

$$
y[n] = -3\delta[n] - 6\delta[n-1] + 3\delta[n-2] + 6\delta[n-3], \quad 0 \le n \le 3
$$

Exercise 5.3.

A signal $x[n]$ has been filtered by a linear time-invariant system with impulse response $h[n]=\delta[n]-\frac{1}{2}\delta[n-1]$. The output of the system is the distorted signal $y[n]$. Theoretically, $x[n]$ can be recovered from $y[n]$ through an inverse filter having a system function equal to the reciprocal of the system function of the distorting filter.

(a) Determine the z-transform $H(z)$ and the four-point DFT $H[k]$ of the impulse response $h[n]$.

(b) Let $H_i(z)$ denote the system function of the inverse filter and let $h_i[n]$ be the corresponding impulse response. Determine $h_i[n]$. Is this an FIR or an IIR filter?

(c) If $x[n]=\delta[n]+\delta[n-1]+\delta[n-2]$, calculate $y[n]=x[n]*h[n]$ by doing the linear convolution directly.

(d) Calculate $y[n]$ of Question (c) by multiplying the four-point DFTs of $x[n]$ and $h[n]$ and performing an inverse DFT.

(e) Determine the z-transform Y(z) and determine the inverse z-transform of $Y(z)\cdot H_i(z)$.

Solutions:

(a)

$$H(z)=1-\frac{1}{2}z^{-1}$$

$$H[k]=1-\frac{1}{2}W_4^1$$

(b)

$$H_i(z)=\frac{1}{1-\frac{1}{2}z^{-1}}$$

$$h_i[n]=(\frac{1}{2})^n u[n]$$

IIR

(c)

$$y[n]=\delta[n]+\frac{1}{2}\delta[n-1]+\frac{1}{2}\delta[n-2]-\frac{1}{2}\delta[n-3]$$

(d)

$$X[k] = 1 + W_4^1 + W_4^2$$

$$Y[k] = 1 + \frac{1}{2}W_4^1 + \frac{1}{2}W_4^2 - \frac{1}{2}W_4^3$$

$$y[n] = \delta[n] + \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n-2] - \frac{1}{2}\delta[n-3]$$

(e)

$$Y(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{2}z^{-3}$$

$$Y(z)H_i(z) = (1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{2}z^{-3})/(1 - \frac{1}{2}z^{-1})$$

$$Y(z)H_i(z) \leftrightarrow \delta[n] + \delta[n-1] + \delta[n-2]$$