Digital Signal Processing, Fall 2010

Lecture 3: Sampling and reconstruction, transform analysis of LTI systems

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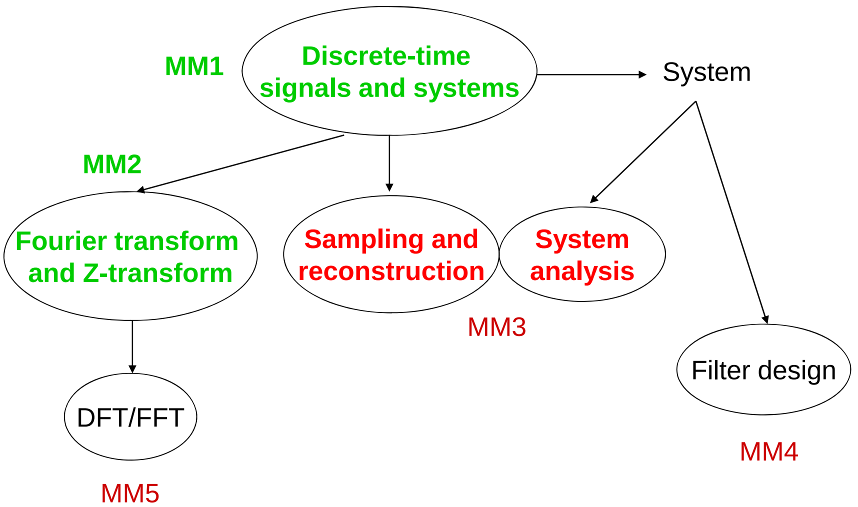
Course at a glance



Part I-A: Periodic sampling

- Part I: sampling and reconstruction
  - Periodic sampling
  - Frequency domain representation of the sampling
  - Reconstruction
- Part II: system analysis

Periodic sampling

- From continuous-time $x_c(t)$ to discrete-time $x[n]$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

  - Sampling period T
  - Sampling frequency $f_s = 1/T$

$$\Omega_s = 2\pi / T$$

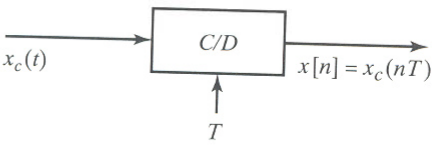


Figure 4.1 Block diagram representation of an ideal continuous-to-discrete-time (C/D) converter.

Two stages

In practice?

- Mathematically
  - Impulse train modulator
  - Conversion of the impulse train to a sequence

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$x_s(t) = x_c(t)s(t)$$

$$= x_c(t)\sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

$$(\; x_c(t) = \int_{-\infty}^{\infty} x_c(\tau)\delta(t-\tau)d\tau \;)$$

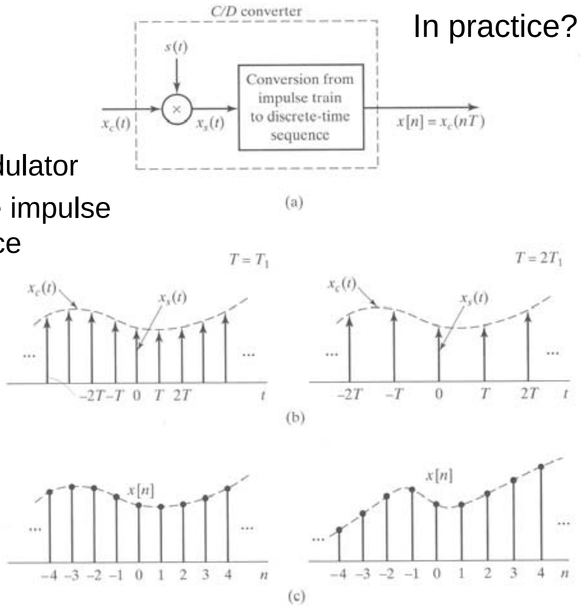


Figure 4.2 Sampling with a periodic impulse train followed by conversion to a discrete-time sequence. (a) Overall system. (b) $x_s(t)$ for two sampling rates. (c) The output sequence for the two different sampling rates.

Periodic sampling

- Tow-stage representation
  - Strictly a mathematical representation that is convenient for gaining insight into sampling in both the time and frequency domains.
  - Physical implementation is different.
  - $x_s(t)$ a continuous-time signal, an impulse train, zero except at nT
  - $x[n]$ a discrete-time sequence, time normalization, no explicit information about sampling rate
- Many-to-many → in general not invertible

Part I-B: Freq. domain represent.

- Part I: sampling and reconstruction
  - Periodic sampling
  - Frequency domain representation of the sampling
  - Reconstruction
- Part II: system analysis

Frequency-domain representation

- From $x_c(t)$ to $x_s(t)$

The Fourier transform of a periodic impulse train is a periodic impulse train.

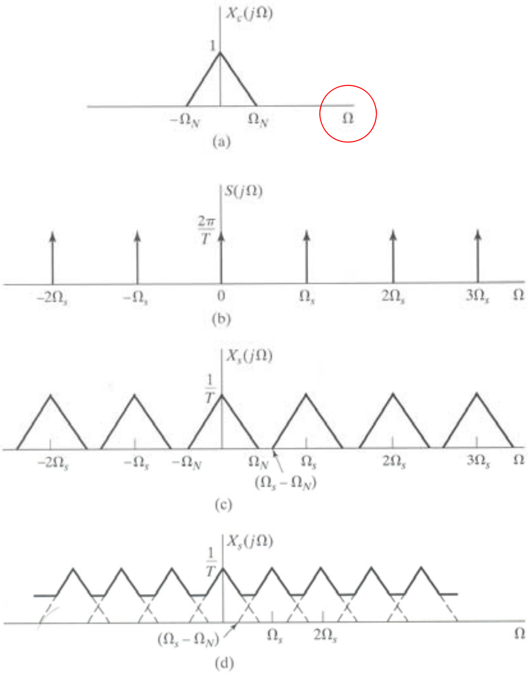
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \quad \leftrightarrow \quad \underset{?}{S(j\Omega)} = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s)$$

$$x_s(t) = x_c(t)s(t) \quad \leftrightarrow \quad X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * S(j\Omega)$$

$$= x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \qquad = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\Omega - k\Omega_s))$$

$$= \sum_{n=-\infty}^{\infty}x_c(nT)\delta(t-nT)$$

- The Fourier transform of $x_s(t)$ consists of periodic repetition of the Fourier transform of $x_c(t)$.



$$S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s)$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\Omega - k\Omega_s))$$

$$\Omega_s - \Omega_N > \Omega_N \quad \text{or} \quad \Omega_s > 2\Omega_N$$

Figure 4.3 Effect in the frequency domain of sampling in the time domain. (a) Spectrum of the original signal. (b) Spectrum of the sampling function. (c) Spectrum of the sampled signal with $\Omega_s > 2\Omega_N$. (d) Spectrum of the sampled signal with $\Omega_s < 2\Omega_N$.

VERSITY

Recovery

$$X_r(j\Omega) = H_r(j\Omega)X_s(j\Omega)$$

Ideal lowpass filter with gain T and cutoff frequency Ω_c

$$\Omega_N < \Omega_c < (\Omega_s - \Omega_N)$$

$$X_r(j\Omega) = X_c(j\Omega)$$

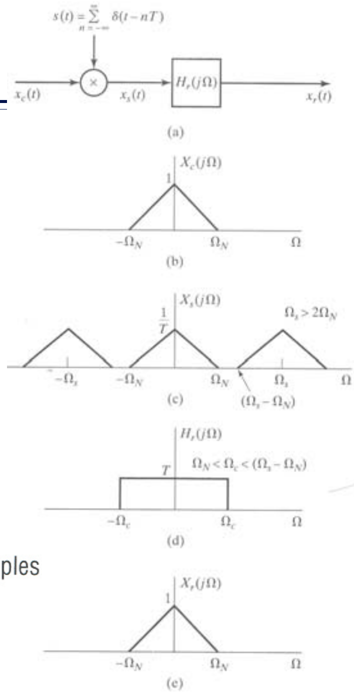
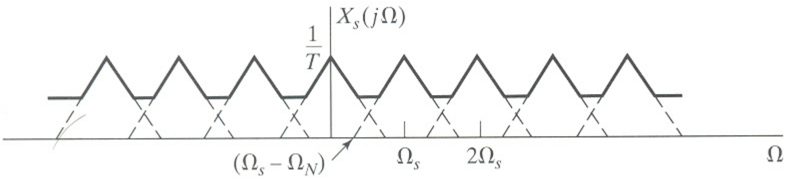


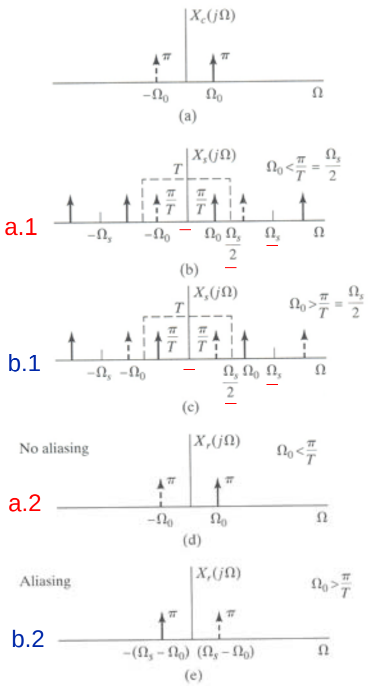
Figure 4.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.

Aliasing distortion

- Due to the overlap among the copies of $X_c(j\Omega)$, due to $\Omega_s \le 2\Omega_N$
- $X_c(j\Omega)$ not recoverable by lowpass filtering



Aliasing – an example



$$x_c(t) = \cos \Omega_0 t$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$x_r(t) = \cos \Omega_0 t$$

$$x_r(t) = \cos(\Omega_s - \Omega_0)t$$

Figure 4.5 The effect of aliasing in the sampling of a cosine signal.

Nyquist sampling theorem

Given bandlimited signal $x_c(t)$ with

$$X_c(j\Omega) = 0, \quad \text{for } |\Omega| \geq \Omega_N$$

Then $x_c(t)$ is uniquely determined by its samples

$$x[n] = x_c(nT), \qquad -\infty < n < \infty$$

If

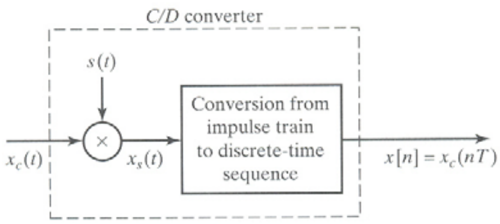
$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$$

Ω_N is called Nyquist frequency

$2\Omega_N$ is called Nyquist rate

Fourier transform of x[n]

From $x_s(t)$ to $x[n]$



$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

From $X_s(j\Omega)$ to $X(e^{j\omega})$

By taking continuous-time Fourier transform of $x_s(t)$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn} \qquad \left(X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \right)$$

By taking discrete-time Fourier transform of $x[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T})$$

Fourier transform of x[n]

From Slide 14, $X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T})$

From Slide 8, $X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$

So, $X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\frac{\omega}{T}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))|_{\Omega=\frac{\omega}{T}}$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi k}{T}\right)$$

- i.e. $X(e^{j\omega})$ is simply a frequency-scaled version of $X_s(j\Omega)$ with $\omega = \Omega T$
- $x_s(t)$ retains a spacing between samples equal to the sampling period T while $x[n]$ always has unity space.

Sampling and reconstruction of Sin Signal

$x_c(t) = \cos(4000\pi t) \rightarrow \Omega_0 = 4000\pi$

$T = 1/6000 \rightarrow \Omega_s = 2\pi/T = 12000\pi \quad \therefore$ no aliasing

$x[n] = x_c(nT) = \cos(4000\pi nT) = \cos((2\pi/3)n) = \cos(\omega_0 n)$

$x_c(t) \leftrightarrow X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi) \qquad X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$

$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T} = X_s(j\omega/T)$ with normalized frequency $\omega = \Omega T$

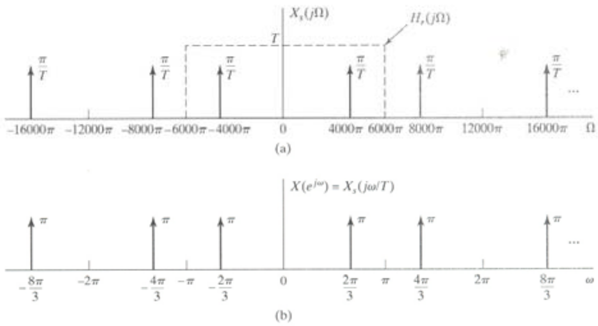


Figure 4.6 Continuous-time (a) and discrete-time (b) Fourier transforms for sampled cosine signal with frequency $\Omega_0 = 4000\pi$ and sampling period $T = 1/6000$.

How about $x_c(t) = \cos(16000\pi t)$

Part I-C: Reconstruction

- Part I: sampling and reconstruction
  - Periodic sampling
  - Frequency domain representation of the sampling
  - Reconstruction
- Part II: system analysis

Requirement for reconstruction

- On the basis of the sampling theorem, samples represent the signal exactly when:
  - Bandlimited signal
  - Enough sampling frequency
  - + knowledge of the sampling period → recover the signal

Reconstruction steps

(1) Given x[n] and T, the impulse train is

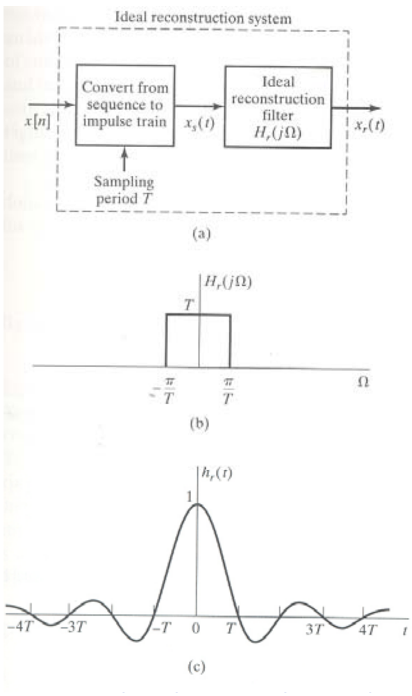
$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$$

i.e. the *n*th sample is associated with the impulse at t=nT.

(2) The impulse train is filtered by an ideal lowpass CT filter with impulse response $h_r(t) \leftrightarrow H_r(j\Omega)$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n) h_r(t-nT)$$

$$X_r(j\Omega) = H_r(j\Omega) X(e^{j\Omega T})$$

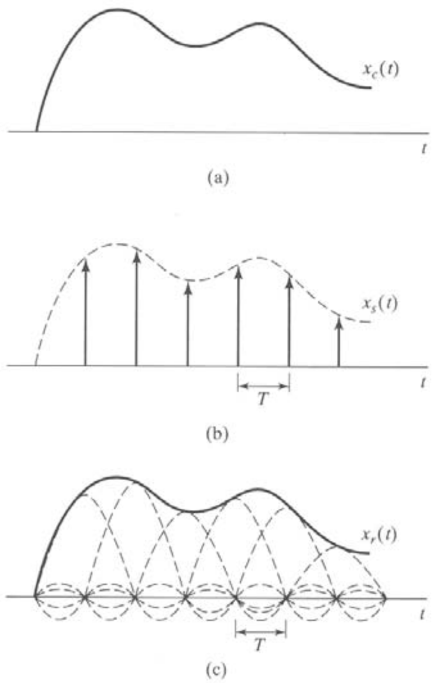


Commonly choose cutoff frequncy as

$$\Omega_c = \Omega_s / 2 = \pi / T$$

$$h_r(t) = \frac{\sin(\pi t / T)}{\pi t / T}$$

Figure 4.8 (a) Block diagram of an ideal bandlimited signal reconstruction system. (b) Frequency response of an ideal reconstruction filter. (c) Impulse response of an ideal reconstruction filter.



CT signal

Modulated impulse train

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$

Figure 4.9 Ideal bandlimited interpolation.

Ideal discrete-to-continuous-time converter

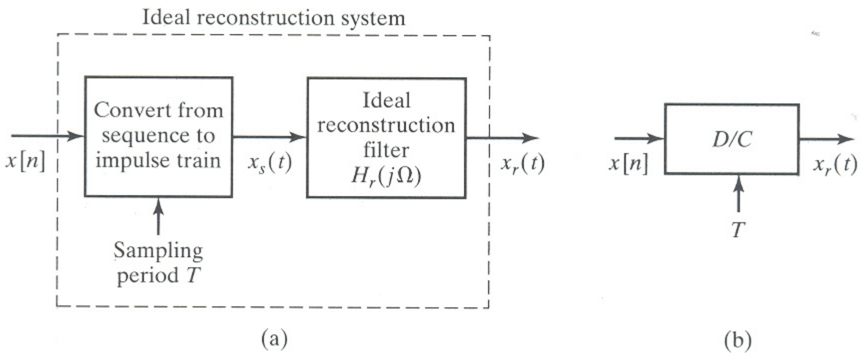
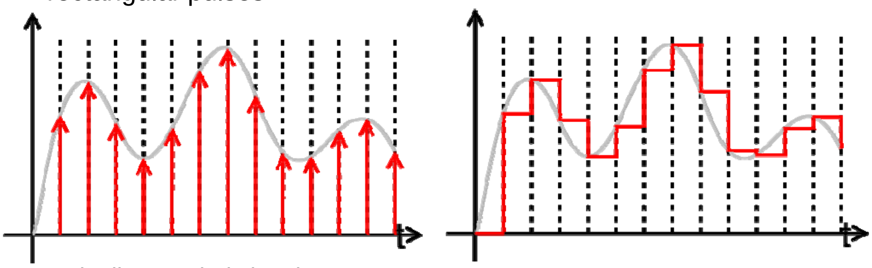


Figure 4.10 (a) Ideal bandlimited signal reconstruction. (b) Equivalent representation as an ideal D/C converter.

discrete-to-continuous-time converter

"Practical DACs do not output a sequence of dirac impulses (that, if ideally low-pass filtered, result in the original signal before sampling) but instead output a sequence of piecewise constant values or rectangular pulses"



Ideally sampled signal.

Piecewise constant signal typical of a practical DAC output.

From http://en.wikipedia.org/wiki/Digital-to-analog_converter.

Applications

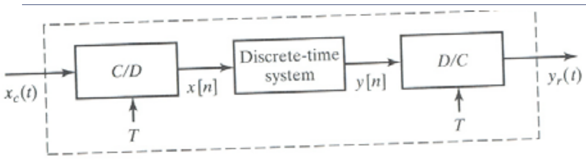


Figure 4.11 Discrete-time processing of continuous-time signals.

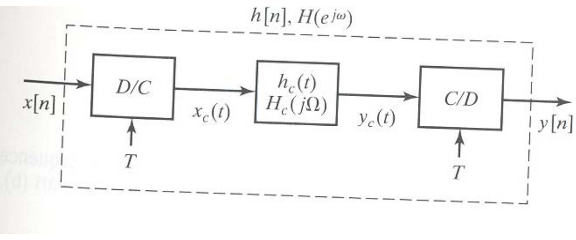


Figure 4.16 Continuous-time processing of discrete-time signals.

Part II System analysis

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

System analysis

- Three domains
  - Time domain: impulse response, convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  - Frequency domain: frequency response

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

  - z-transform: system function

$$Y(z) = X(z)H(z)$$

- LTI system is completed characterized by …

Part II-A: Frequency response

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

Frequency response

- Relationship btw Fourier transforms of input and output

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- In polar form
  - Magnitude → magnitude response, gain, distortion

$$|Y(e^{j\omega})| = |X(e^{j\omega})| \cdot |H(e^{j\omega})|$$

  - Phase → phase response, phase shift, distortion

$$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$$

Ideal lowpass filter – an example

- Frequency response

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

  - Frequency selective filter
- Impulse response

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty \qquad\qquad h[n] \overset{?}{=} 0, \quad n < 0$$

  - Noncausal, cannot be implemented!
  - How to make a noncausal system causal?
    - In general, any noncausal FIR system can be made causal by cascading it with a sufficiently long delay!
    - But ideal lowpass filter is an IIR system!

Phase distortion and delay

- Ideal delay system

$$h_{id}[n] = \delta[n - n_d]$$

Delay distortion

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

$$|H_{id}(e^{j\omega})| = 1$$

$$\angle H_{id}(e^{j\omega}) = -\omega n_d, |\omega| < \pi$$

Linear phase distortion

- Ideal lowpass filter with linear phase

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

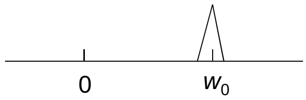
$$h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \qquad -\infty < n < \infty$$

Ideal lowpass filter is always noncausal!

Group delay

- A measure of the linearity of the phase
- Concerning the phase distortion on a narrowband signal

$$x[n] = s[n]\cos(\omega_0 n)$$



- For this input with spectrum only around w_0, phase effect can be approximated around w_0 as the linear approximation (though in reality maybe nonlinear)

$$\angle H(e^{j\omega}) \approx -\omega n_d - \phi_0$$

and the output is approximately

$$y[n] \approx |H(e^{j\omega_0})|\, s[n-n_d]\cos(\omega_0(n-n_d)-\phi_0)$$

- Group delay $\tau = grd[H(e^{j\omega})] = -\dfrac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$

An example of group delay

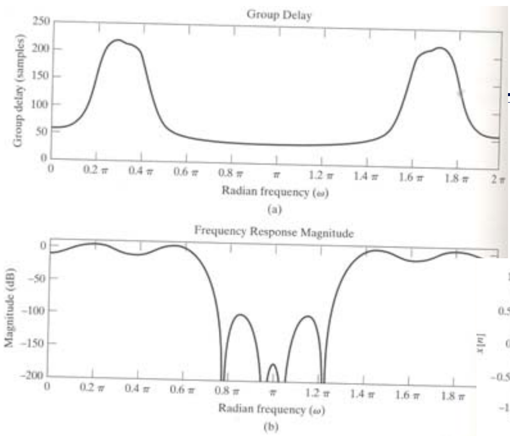


Figure 5.1 Frequency response magnitude and group delay for the filter in Example 5.1.

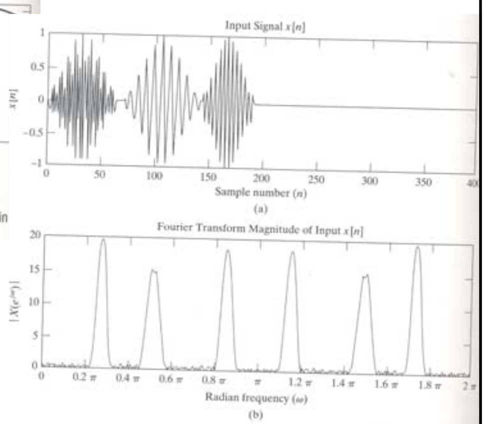


Figure 5.2 Input signal and associated Fourier transform magnitude for Example 5.1.

An example of group delay

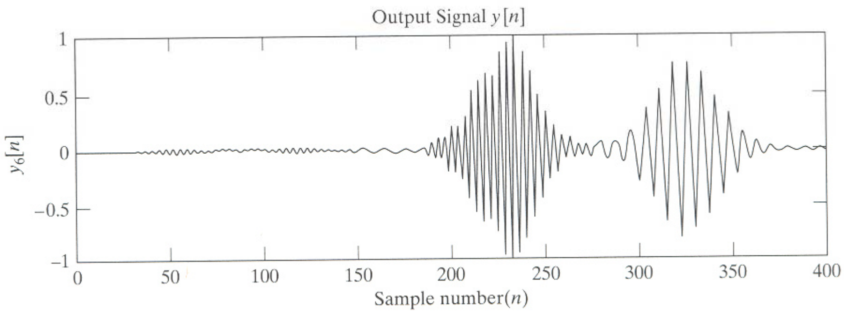


Figure 5.3 Output signal for Example 5.1.

Since the filter has considerable attenuation at $\omega = 0.85\pi$, the pulse at that frequency is not clearly present in the output. Also, since the group delay at $\omega = 0.25\pi$ is approximately 200 samples and at $\omega = 0.5\pi$ is approximately 50 samples, the second pulse in $x[n]$ will be delayed by about 200 samples and the third pulse by 50 samples, as we see is the case in Figure 5.3.

Part II-B: System functions

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

System function of LCCDE systems

- Linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- z-transform format

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \left(\frac{b_0}{a_0}\right) \frac{\prod_{m=1}^{M} (1 - c_m z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$(1 - c_m z^{-1})$ in the numerator

a zero at $z = c_m$ a pole at $z = 0$

$(1 - d_k z^{-1})$ in the denominator

a zero at $z = 0$ a pole at $z = d_k$

Stability and causality

- Stable
  - h[n] absolutely summable
  - H(z) has a ROC including the unit circle
- Causal
  - h[n] right side sequence
  - H(z) has a ROC being outside the outermost pole

Inverse systems

- Many systems have inverses, specially systems with rational system functions

$$G(z) = H(z)H_i(z) = 1$$

$$H_i(z) = \frac{1}{H(z)}$$

$$g[n] = h[n] * h_i[n] = \delta[n]$$

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{m=1}^{M}(1 - c_m z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$H_i(z) = \left(\frac{a_0}{b_0}\right)\frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{m=1}^{M}(1 - c_m z^{-1})}$$

  - Poles become zeros and vice versa.
  - ROC: must have overlap btw the two for the sake of G(z).

Example

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}, |z| > 0.9$$

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}} = \frac{1}{1 - 0.5z^{-1}} - \frac{0.9z^{-1}}{1 - 0.5z^{-1}}$$

So,

$$|z| > 0.5$$

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1]$$

Part II-C: All-pass systems

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

All-pass systems

- Consider the following stable system function

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$

$$= e^{-j\omega} \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

- $|H_{ap}(e^{j\omega})| = 1$ all-pass system: for which the frequency response magnitude is a constant.

Example: First-order all-pass system

P275 Example 5.13

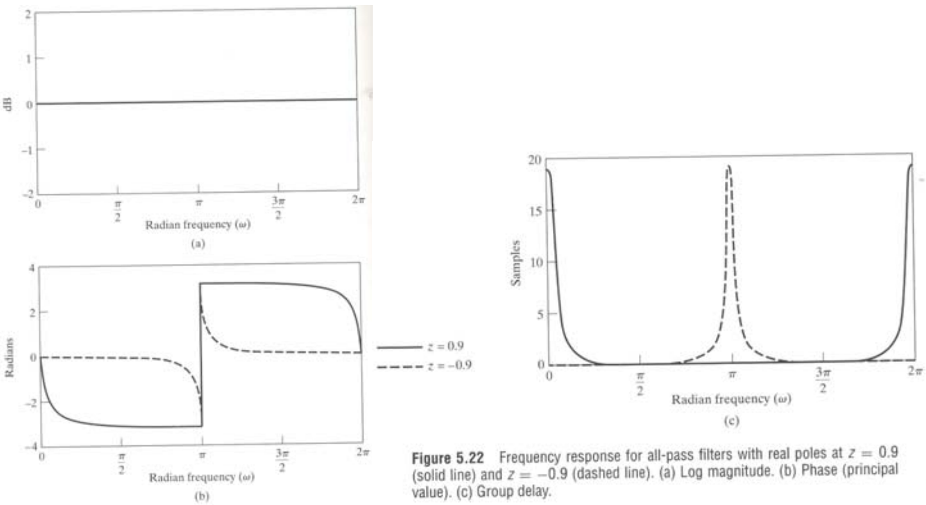


Figure 5.22 Frequency response for all-pass filters with real poles at $z = 0.9$ (solid line) and $z = -0.9$ (dashed line). (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

Part II-D: Minimum-phase systems

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

Minimum-phase systems

- Magnitude does not uniquely characterize the system
  - Stable and causal → poles inside unit circle, no restriction on zeros
  - Zeros are also inside unit circle → inverse system is also stable and causal (in many situations, we need inverse systems!)
  - → such systems are called minimum-phase systems (explanation to follow): are stable and causal and have stable and causal inverses

Minimum-phase and all-pass decomposition

Any rational system function can be expressed as:

$$H(z) = H_{\min}(z)H_{ap}(z)$$

Suppose H(z) has one zero outside the unit circle at

$z = 1/c^*, |c| < 1$

$$H(z) = H_1(z)(z^{-1} - c^*)$$

$$= \underbrace{H_1(z)(1 - cz^{-1})}_{\text{minimum-phase}} \underbrace{\frac{z^{-1} - c^*}{1 - cz^{-1}}}_{\text{all-pass}}$$

minimum-phase all-pass

Frequency response compensation

When the distortion system is not minimum-phase system:

$$H_d(z) = H_{d\min}(z)H_{ap}(z) \qquad H_c(z) = \frac{1}{H_{d\min}(z)}$$

$$G(z) = H_d(z)H_c(z) = H_{ap}(z)$$

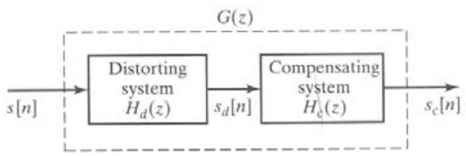


Figure 5.25 Illustration of distortion compensation by linear filtering.

Frequency response magnitude is compensated

Phase response is the phase of the all-pass

Properties of minimum-phase systems

- From minimum-phase and all-pass decomposition

$$H(z) = H_{\min}(z)H_{ap}(z)$$

$$\arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

- From the previous figure, the continuous-phase curve of an all-pass system is negative for $0 \le \omega \le \pi$
- So change from minimum-phase to non-minimum-phase (+all-pass phase) always decreases the continuous phase or increases the negative of the phase (called the phase-lag function). Minimum-phase is more precisely called minimum phase-lag system

Part II-E: GLP systems

- Part I: sampling and reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

Design a system with non-zero phase

- System design sometimes desires
  - Constant frequency response magnitude
  - Zero phase, when not possible
    - accept phase distortion, in particular linear phase since it only introduce time shift
    - Nonlinear phase will change the shape of the input signal though having constant magnitude response

Ideal delay

$$H_{id}(e^{j\omega}) = e^{-j\omega\alpha}, |\omega| < \pi$$

$$|H_{id}(e^{j\omega})| = 1$$

$$\angle H_{id}(e^{j\omega}) = -\omega\alpha, |\omega| < \pi$$

$$grd[H_{id}(e^{j\omega})] = \alpha$$

$$h_{id}[n] = \frac{\sin \pi(n-\alpha)}{\pi(n-\alpha)}$$

when $\alpha = n_d$

$$h_{id}[n] = \delta[n - n_d]$$

Ideal lowpass with linear phase

$$h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi(n - n_d)}$$

Generalized linear phase

- Linear phase filters

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}$$

- Generalized linear phase filters

$$H(e^{j\omega}) = A(e^{j\omega}) e^{-j\omega\alpha + j\beta}$$

$A(e^{j\omega})$ is a real function of ω,

α and β are real constants

Summary

- Part I: sampling and reconstruction
  - Periodic sampling
  - Frequency domain representation
  - Reconstruction
- Part II: system analysis
  - Frequency response
  - System functions
  - All-pass systems
  - Minimum-phase systems
  - Linear systems with generalized linear phase

Course at a glance