Digital Signal Processing

http://kom.aau.dk/~zt/cources/Digital_signal_processing/

Solutions 2 (MM2)

2.1 $X(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}}$, *with* $-1 < a < 0$.

Solution:

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} = \frac{1}{1-a\cos\omega + ja\sin\omega} = \frac{1-a\cos\omega - ja\sin\omega}{1-2a\cos\omega + a^2}$$

a)

$$|X(e^{j\omega})| = \sqrt{\frac{1}{1-2a\cos\omega + a^2}}$$

b)

$$\angle X(e^{j\omega}) = \arctan(\frac{-a\sin\omega}{1-a\cos\omega})$$

Exercise 2.2

Solution:

a) $X(z) = \sum_{n=0}^{\infty}(\frac{1}{2})^n z^{-n} = \dfrac{z}{z-\frac{1}{2}}$

For convergence of $X(z)$, require that $\sum_{n=0}^{\infty}|\frac{1}{2}z^{-1}|^n < \infty$.

So, ROC is $|z| > \frac{1}{2}$.

b) $X(z) = -\sum_{n=-\infty}^{-1}(½)^n z^{-n} = \dfrac{z}{z-\frac{1}{2}}, \quad |z| < \frac{1}{2}$.

c) $X(z) = z^{-1}Z(\delta[n]) = z^{-1}, \quad |z| > 0$.

Exercise 2.3. The input to a LTI-system is $u[n]$ and the output $y[n]$ is

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1].$$

Find the transfer function, H(z).

Solution:

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1.$$

$$Y(z) = \frac{4z}{1 - \frac{1}{2} z^{-1}}, \quad |z| > \frac{1}{2}.$$

$$H(z) = \frac{4z(1 - z^{-1})}{1 - \frac{1}{2} z^{-1}}, |z| > \frac{1}{2}$$

Exercise 2.4. Given the Z-transform $X(z) = \dfrac{1 - \frac{1}{2} z^{-1}}{(1 + \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2})}, |z| > \frac{1}{2}$. Try using partial fraction expansion to find x[n].

Solution:

$$X(z) = \frac{1 - \frac{1}{2} z^{-1}}{1 + \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}, \qquad |z| > \frac{1}{2}$$

Partial fractions:

$$X(z) = \frac{-3}{1 + \frac{1}{4} z^{-1}} + \frac{4}{1 + \frac{1}{2} z^{-1}}, \qquad |z| > \frac{1}{2}$$

$$x[n] = \left[-3\left(-\frac{1}{4} \right)^n + 4\left(-\frac{1}{2} \right)^n \right] u[n]$$

Exercise 2.5. Given the transfer function $H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$ of a causal LTI system. Is the system stable ? (First find the ROC)

Solution:

$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$, causal and LTI,

Since system is causal, ROC is $|z| > \frac{1}{2}$, which includes the unit circle, the system is stable.