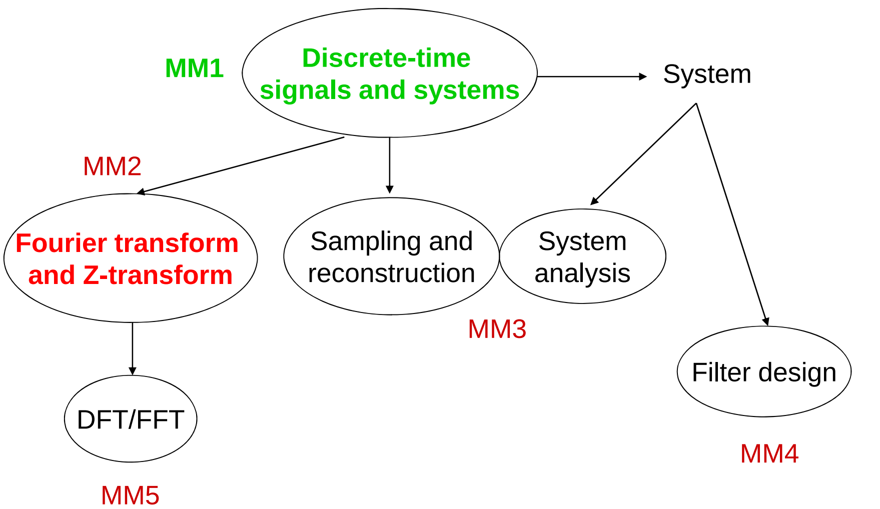
Digital Signal Processing, Fall 2010

Lecture 2: Fourier transform, frequency response, and Z-transform

Zheng-Hua Tan

Department of Electronic Systems
Aalborg University, Denmark
zt@es.aau.dk

Course at a glance



Part I-A: Fourier representation

- Fourier transform and frequency response
  - Frequency-domain representation of discrete-time signals and systems
  - Fourier transform theorems
- Z-transform

Signal representation & system response

- Impulse response of LTI systems

$$x[n] = \delta[n] \rightarrow y[n] = h[n]$$

- A sum of scaled, delayed impulse

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \implies y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n]*h[n]$$

Convolution sum!

System response to complex exponentials

Complex exponentials as input to LTI system *h*[*n*]

$$x[n] = e^{j\omega n}, \quad -\infty < n < \infty$$

$$y[n] = T\{e^{j\omega n}\} = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

Define $$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

Then $$y[n] = H(e^{j\omega}) e^{j\omega n}$$

Frequency and impulse responses are a Fourier transform pair

→ signal representation based on complex exponentials, i.e. Fourier representation.

Fourier representation

- Fourier representation of a sequence: Fourier transform and the inverse Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- FT "is an operation that transforms one function (typically in the time domain) into another (in frequency domain). So the FT is often called the frequency domain representation of the original function & describes which frequencies are present in the original function. The term FT refers both to the frequency domain representation and to the process or formula that "transforms" one function into the other."
- "Fourier transform defines a relationship between a signal in the time domain and its representation in the frequency domain. … The original signal can be recovered from knowing the Fourier transform, and vice versa."

System input and output

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(e^{j\omega}) \cdot e^{j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \rightarrow y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) \cdot H(e^{j\omega}) \cdot e^{j\omega n}d\omega$$

$$\text{Input: } X(e^{j\omega}) \rightarrow \text{Output: } X(e^{j\omega}) \cdot H(e^{j\omega})$$

$$x[n] = \delta[n] \Longrightarrow y[n] = h[n]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \Longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Frequency response

- The frequency response of discrete-time LTI systems is always a periodic function of the frequency variable w with period 2π .

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}e^{-j2\pi n} = H(e^{j\omega})$$

  - Only specify over the interval $-\pi < \omega \le \pi$
  - The 'low frequencies' are close to 0
  - The 'high frequencies' are close to $\pm\pi$

Frequency response

- Frequency response is generally complex

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

$$= |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$ describes changes in magnitude and phase

- Frequency response of the ideal delay system

$$y[n] = x[n - n_d] \rightarrow h[n] = \delta[n - n_d]$$

Calculate Fourier transform $\rightarrow H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n} = e^{-j\omega n_d}$

$$H_R(e^{j\omega}) = \cos(\omega n_d), \qquad H_I(e^{j\omega}) = -\sin(\omega n_d).$$

$$|H(e^{j\omega})| = 1, \qquad \angle H(e^{j\omega}) = -\omega n_d.$$

Ideal frequency-selective filters

- For which the frequency response is unity over a certain range of frequencies, and is zero at the remaining frequencies.
  - Ideal low-pass filter: passes only low and rejects high

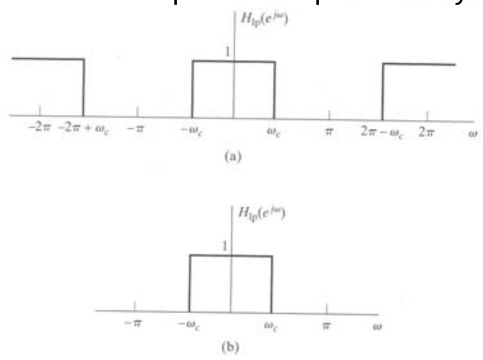


Figure 2.17 Ideal lowpass filter showing (a) periodicity of the frequency response and (b) one period of the periodic frequency response.

Sufficient condition for Fourier transform

- Condition for the convergence of the infinite sum

$$|X(e^{j\omega})| = |\sum_{-\infty}^{\infty} x[n]e^{-j\omega n}|$$

$$\leq \sum_{-\infty}^{\infty} |x[n]||e^{-j\omega n}|$$

$$\leq \sum_{-\infty}^{\infty} |x[n]| < \infty$$

- If x[n] is absolutely summable, its Fourier transform exists (sufficient condition).

Example 1: Exponential sequence

- Let $x[n] = a^n u[n]$

- The Fourier transform of it

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1 - ae^{-j\omega}} \quad \text{if } |a| < 1$$

Example 2: ideal lowpass filter

- Frequency response $H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

  - Noncausal.
  - Not absolutely summable.

$$H_{lp}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

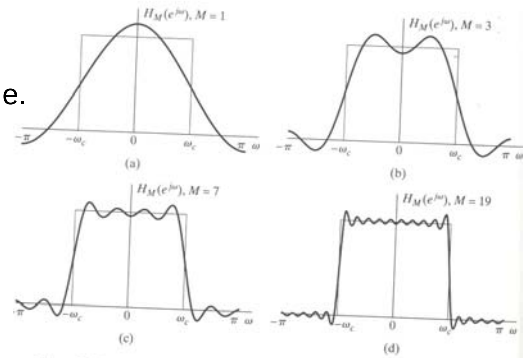


Figure 2.21 Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.

Example 3: a constant sequence

- Constant sequence

$$x[n] = 1$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

where $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$

- Its Fourier transform is defined as the periodic impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

Part I-B: Fourier transform theorems

- Fourier transform and frequency response
  - Frequency-domain representation of discrete-time signals and systems
  - Fourier transform theorems
- Z-transform

Linearity of the Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$x_1[n] \overset{F}{\leftrightarrow} X_1(e^{j\omega})$$

$$x_2[n] \overset{F}{\leftrightarrow} X_2(e^{j\omega})$$

$$ax_1[n] + bx_2[n] \overset{F}{\leftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Time shifting and frequency shifting

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$x[n-n_d] \overset{F}{\leftrightarrow} e^{-j\omega n_d} X(e^{j\omega})$$

$$e^{j\omega_0 n}x[n] \overset{F}{\leftrightarrow} X(e^{j(\omega-\omega_0)})$$

Time reversal

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$x[-n] \overset{F}{\leftrightarrow} X(e^{-j\omega})$$

The convolution theorem

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$h[n] \overset{F}{\leftrightarrow} H(e^{j\omega})$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n]*h[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

The modulation or Windowing theorem

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$w[n] \overset{F}{\leftrightarrow} W(e^{j\omega})$$

$$y[n] = x[n]w[n]$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n - n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\dfrac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{(1 - ae^{-j\omega})^2}$
7. $\dfrac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\lvert r\rvert < 1)$	$\dfrac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

Example

- Determining a Fourier transform using Tables 2.2 and 2.3

$$x[n] = a^n u[n-5]$$

- Solution

$$x_1[n] = a^n u[n] \overset{F}{\leftrightarrow} X_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$$x_2[n] = x_1[n-5] \qquad (\text{i.e.} = a^{n-5}u[n-5])$$

$$X_2(e^{j\omega}) = e^{-j5\omega}X_1(e^{j\omega}) = \frac{e^{-j5\omega}}{1-ae^{-j\omega}}$$

$$x[n] = a^5 x_2[n] \qquad (\text{i.e.} = a^n u[n-5])$$

$$X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1-ae^{-j\omega}}$$

Part II-A: Z-transform

- Fourier transform and frequency response
- z-transform
  - Z-transform
  - Properties of the ROC
  - Inverse z-transform
  - Properties of z-transform

Limitation of Fourier transform

- Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Condition for the convergence of the infinite sum

$$|X(e^{j\omega})| = |\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}| \le \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| \le \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- If x[n] is absolutely summable, its Fourier transform exists (sufficient condition).
- Example $x[n] = a^n u[n]$ $\quad |a|<1:\ X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$

$$a=1:\ X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$$

$|a|>1:$ No

z-transform

- Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

- z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad x[n] \overset{Z}{\leftrightarrow} X(z)$$

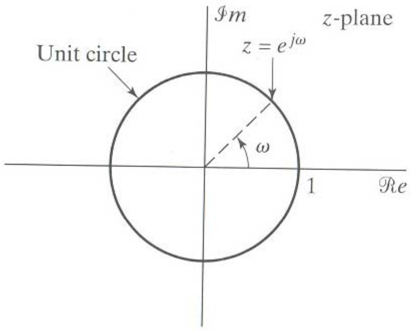
- The complex variable z in polar form $z = re^{j\omega}$

$$X(z) = X(re^{j\omega}) = \sum_{-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

$$|z| = r = 1, \quad X(z) = X(e^{j\omega})$$

z-plane

- z-transform is a function of a complex variable → using the complex z-plane



Z-transform on unit circle <-> Fourier transform

Linear frequency axis in Fourier transform
→Unit circle in z-transform (periodicity in freq. of Fourier transform)

Figure 3.1 The unit circle in the complex z-plane.

Region of convergence – ROC

- Fourier transform does not converge for all sequences

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- z-transform does not converge for all sequences or for all values of z.

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \; z^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

- ROC – for any given seq., the set of values of z for which the z-transform converges

$$\sum_{n=-\infty}^{\infty} | x[n]r^{-n} | < \infty \qquad \sum_{n=-\infty}^{\infty} | x[n] | \, | z |^{-n} < \infty$$

ROC is ring!

ROC

- Outer boundary is a circle (may extend to infinity)
- Inner boundary is a circle (may extend to incl. origin)
- If ROC includes unit circle, Fourier transform converges

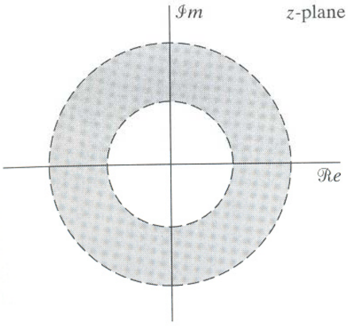


Figure 3.2 The region of convergence (ROC) as a ring in the *z*-plane. For specific cases, the inner boundary can extend inward to the origin, and the ROC becomes a disc. For other cases, the outer boundary can extend outward to infinity.

Zeros and poles

- The most important and useful z-transforms – rational function:

$$X(z) = \frac{P(z)}{Q(z)}$$

$P(z)$ and $Q(z)$ are polynomials in z

- Zeros: values of *z* for which X(z)=0.
- Poles : values of *z* for which X(z) is infinite.
- Close relation between poles and ROC

Right-sided exponential sequence

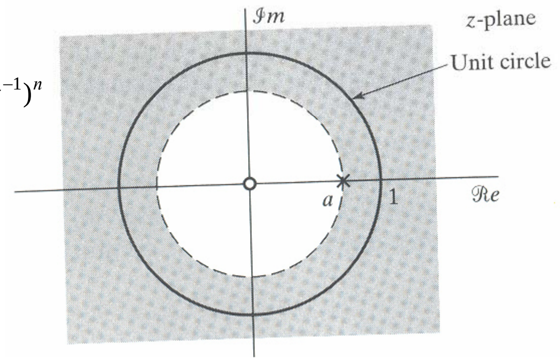
$x[n] = a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

- ROC

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

- z-transform



$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|$$

$$u[n] \overset{Z}{\leftrightarrow} \frac{1}{1 - z^{-1}}, \qquad |z| > 1$$

Left-sided exponential sequence

$x[n] = -a^n u[-n-1]$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

- ROC

$$\sum_{n=0}^{\infty} |a^{-1} z|^n < \infty$$

- z-transform

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| < |a|$$

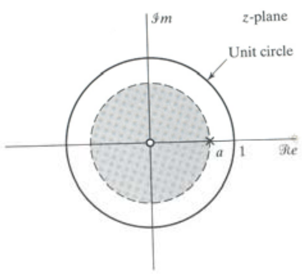


Figure 3.4 Pole–zero plot and region of convergence for Example 3.2.

Sum of two exponential sequence

$$x[n] = (\frac{1}{2})^n u[n] + (-\frac{1}{3})^n u[n]$$

$$X(z) = \sum_{n=0}^{\infty} (\frac{1}{2} z^{-1})^n + \sum_{n=0}^{\infty} (-\frac{1}{3} z^{-1})^n$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

- ROC

$$|z| > \frac{1}{2} \text{ and } |z| > \frac{1}{3}$$

$$|z| > \frac{1}{2}$$

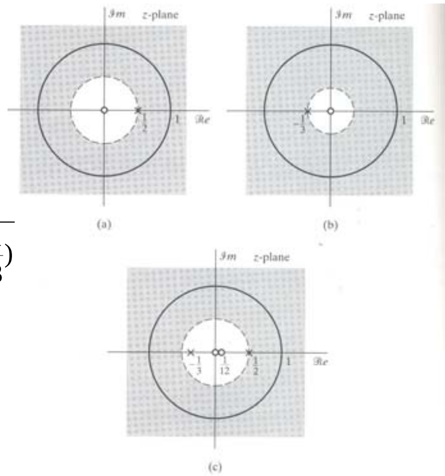


Figure 3.5 Pole–zero plot and region of convergence for the individual terms and the sum of terms in Examples 3.3 and 3.4. (a) $1/(1 - \frac{1}{2}z^{-1})$, $|z| > \frac{1}{2}$. (b) $1/(1 + \frac{1}{3}z^{-1})$, $|z| > \frac{1}{3}$. (c) $1/(1 - \frac{1}{2}z^{-1}) + 1/(1 + \frac{1}{3}z^{-1})$, $|z| > \frac{1}{2}$.

Sum of two exponential sequence

Another way to calculate:

$$x[n] = (\frac{1}{2})^n u[n] + (-\frac{1}{3})^n u[n]$$

$$(\frac{1}{2})^n u[n] \stackrel{Z}{\leftrightarrow} \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad |z| > \frac{1}{2}$$

$$(-\frac{1}{3})^n u[n] \stackrel{Z}{\leftrightarrow} \frac{1}{1 + \frac{1}{3} z^{-1}}, \quad |z| > \frac{1}{3}$$

$$(\frac{1}{2})^n u[n] + (-\frac{1}{3})^n u[n] \stackrel{Z}{\leftrightarrow} \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} \quad |z| > \frac{1}{2}$$

Two-sided exponential sequence

$$x[n] = (-\frac{1}{3})^n u[n] - (\frac{1}{2})^n u[-n-1]$$

$$(-\frac{1}{3})^n u[n] \overset{Z}{\leftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$-(\frac{1}{2})^n u[-n-1] \overset{Z}{\leftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{3}, \ |z| < \frac{1}{2}$$

$$= \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

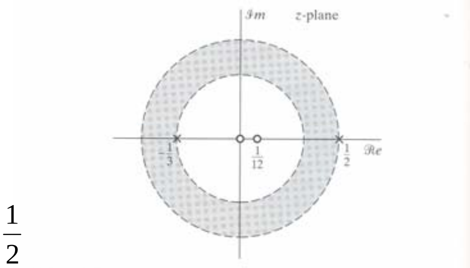


Figure 3.6 Pole–zero plot and region of convergence for Example 3.5.

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $[\cos \omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Part II-B: Properties of the ROC

- Fourier transform and frequency response
- z-transform
  - Z-transform
  - Properties of the ROC
  - Inverse z-transform
  - Properties of z-transform

PROPERTY 1: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.

PROPERTY 2: The Fourier transform of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If $x[n]$ is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \le n \le N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

PROPERTY 5: If $x[n]$ is a *right-sided sequence*, i.e., a sequence that is zero for $n < N_1 < \infty$, the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

PROPERTY 6: If $x[n]$ is a *left-sided sequence*, i.e., a sequence that is zero for $n > N_2 > -\infty$, the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

Properties of the ROC

- The algebraic expression or pole-zero pattern does not completely specify the z-transform of a sequence → the ROC must be specified!
- Stability, causality and the ROC

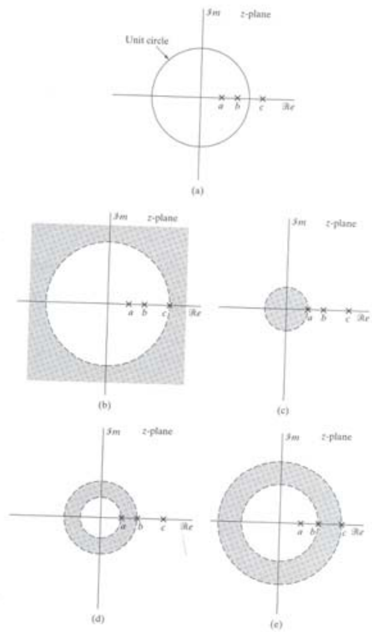


Figure 3.10 Examples of four z- transforms with the same pole–zero locations, illustrating the different possibilities for the region of convergence. Each ROC corresponds to a different sequence: (b) to a right-sided sequence, (c) to a left-sided sequence, (d) to a two-sided sequence, and (e) to a two-sided sequence.

Part II-C: Inverse Z-transform

- Fourier transform and frequency response
- z-transform
  - Z-transform
  - Properties of the ROC
  - Inverse z-transform
  - Properties of z-transform

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Inverse z-transform

- Needed for system analysis: 1) z-transform, 2) manipulation, 3) inverse z-transform.

- Approaches:
  - Inspection method
  - Partial fraction expansion

Inspection method

- By inspection, e.g.

$$X(z) = \left(\frac{1}{1-\frac{1}{2}z^{-1}}\right), \qquad |z| > \frac{1}{2}$$

- Make use of

$$a^n u(n) \overset{Z}{\leftrightarrow} \left(\frac{1}{1-az^{-1}}\right), \qquad |z| > |a|$$

$$\therefore \qquad x[n] = \left(\frac{1}{2}\right)^n u[n]$$

if $|z| < \frac{1}{2}$? of course, $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$

Partial fraction expansion

- For rational function, get the format of a sum of simpler terms, and then use the inspection method.

Second-order z-transform

$$X(z)=\frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}},\quad |z|>\frac{1}{2}$$

$$X(z)=\frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$X(z)=\frac{A_1}{(1-\frac{1}{4}z^{-1})}+\frac{A_2}{(1-\frac{1}{2}z^{-1})}$$

$$A_1=(1-\frac{1}{4}z^{-1})X(z)|_{z=1/4}=-1$$

$$A_2=(1-\frac{1}{2}z^{-1})X(z)|_{z=1/2}=2$$

$$X(z)=\frac{-1}{(1-\frac{1}{4}z^{-1})}+\frac{2}{(1-\frac{1}{2}z^{-1})}$$

$$x[n]=2(\frac{1}{2})^n u[n]-(\frac{1}{4})^n u[n]$$

$$X(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$X(z)=\frac{b_0}{a_0}\frac{\sum_{k=1}^{M}(1-c_k z^{-1})}{\sum_{k=1}^{N}(1-d_k z^{-1})}$$

$$X(z)=\sum_{k=1}^{N}\frac{A_k}{1-d_k z^{-1}}$$

$$A_k=X(z)(1-d_k z^{-1})|_{z=d_k}$$

What about M>=N?

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}},\quad |z|>1 \qquad X(z)=\frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

$$X(z)=B_0+\frac{A_1}{(1-\frac{1}{2}z^{-1})}+\frac{A_2}{(1-z^{-1})}$$

$$\begin{array}{r} 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 \overline{\big)\, z^{-2}+2z^{-1}+1} \\ \underline{z^{-2}-3z^{-1}+2} \\ -5z^{-1}-1 \end{array}$$

$B_0=2$ Found by long division.

$$A_1=(1-\frac{1}{2}z^{-1})X(z)|_{z=1/2}=-9$$

$$A_2=(1-z^{-1})X(z)|_{z=1}=8$$

$$X(z)=2-\frac{9}{(1-\frac{1}{2}z^{-1})}+\frac{8}{(1-z^{-1})}$$

$$x[n]=2\delta[n]-9(\frac{1}{2})^n u[n]+8u[n]$$

Finite-length sequence

$$X(z)=z^2(1-\frac{1}{2}z^{-1})(1+z^{-1})(1-z^{-1})$$

$$X(z)=z^2-\frac{1}{2}z-1+\frac{1}{2}z^{-1}$$

$$x[n]=\delta[n+2]-\frac{1}{2}\delta[n+1]-\delta[n]+\frac{1}{2}\delta[n-1]$$

Part II-D: Properties of Z-transform

- Fourier transform and frequency response
- z-transform
  - Z-transform
  - Properties of the ROC
  - Inverse z-transform
  - Properties of z-transform

Linearity

$$x_1[n] \overset{Z}{\leftrightarrow} X_1(z), \quad \text{ROC} = R_{x_1}$$

$$x_2[n] \overset{Z}{\leftrightarrow} X_2(z), \quad \text{ROC} = R_{x_2}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\ z^{-n}$$

- Linearity

$$ax_1[n] + bx_2[n] \overset{Z}{\leftrightarrow} aX_1(z) + bX_2(z), \quad \underline{\text{ROC contains } R_{x_1} \cap R_{x_2}}$$

at least

$$u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$u[n-1] \overset{Z}{\leftrightarrow} \frac{z^{-1}}{1-z^{-1}}, \quad |z| > 1$$

$$u[n] - u[n-1] = \delta[n] \overset{Z}{\leftrightarrow} \frac{1-z^{-1}}{1-z^{-1}} = 1, \text{ All } z$$

Time shifting

$$x[n-n_0] \overset{Z}{\leftrightarrow} z^{-n_0}X(z),$$
$$\text{ROC} = R_{x_1} \text{ (except for 0 or } \infty)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\ z^{-n}$$

- Example

$$X(z) = \frac{1}{z-\frac{1}{4}}, \quad |z| > \frac{1}{4}$$

$$X(z) = \frac{z^{-1}}{1-\frac{1}{4}z^{-1}} = z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$

$$\left(\frac{1}{4}\right)^n u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-\frac{1}{4}z^{-1}}$$

$$\left(\frac{1}{4}\right)^{n-1} u[n-1] \overset{Z}{\leftrightarrow} z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$

Multiplication by exponential sequence

$$z_0^n x[n] \overset{Z}{\leftrightarrow} X(z/z_0), \quad \text{ROC} = |z_0| R_x$$
$$e^{j\omega_0 n} x[n] \overset{F}{\leftrightarrow} X(e^{j(\omega-\omega_0)})$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\ z^{-n}$$

- Examples

$$u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

$$a^n u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-(z/a)^{-1}} = \frac{1}{1-az^{-1}}, \qquad |z| > a$$

$$x[n] = \cos(\omega_0 n) u[n] = \frac{1}{2}(e^{j\omega_0})^n u[n] + \frac{1}{2}(e^{-j\omega_0})^n u[n]$$

$$X(z) = \frac{\frac{1}{2}}{1-e^{j\omega_0}z^{-1}} + \frac{\frac{1}{2}}{1-e^{-j\omega_0}z^{-1}} = \frac{(1-\cos\omega_0 z^{-1})}{1-2\cos\omega_0 z^{-1} + z^{-2}}, \qquad |z| > 1$$

Time reversal

$$x[-n] \overset{Z}{\leftrightarrow} X(1/z),\ \text{ROC} = 1/R_x$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\ z^{-n}$$

- Example

$$x[n] = a^{-n}u[-n]$$
$$X(z) = \frac{1}{1-az},\quad |z| < |a^{-1}|$$

$$x[n] = a^{n}u[n]$$
$$X(z) = \frac{1}{1-az^{-1}},\quad |z| > |a|$$

Convolution of sequences

$$x_1[n] * x_2[n] \overset{Z}{\leftrightarrow} X_1(z)X_2(z),$$
ROC contains $R_{x_1} \cap R_{x_2}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\ z^{-n}$$

- Example

$$x[n] = (\frac{1}{2})^n u[n]$$
$$h[n] = u[n]$$
$$y[n]?$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}},\quad |z| > \frac{1}{2}$$
$$H(z) = \frac{1}{1-z^{-1}},\quad |z| > 1$$
$$Y(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})},\quad |z| > \frac{1}{2}$$
$$= \frac{1}{1-\frac{1}{2}}\left(\frac{1}{(1-z^{-1})} - \frac{\frac{1}{2}}{(1-\frac{1}{2}z^{-1})}\right)$$

$$y[n] = \frac{1}{1-\frac{1}{2}}(u[n] - (\frac{1}{2})^{n+1}u[n])$$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0 \rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
10	3.4.8	Initial-value theorem: $x[n] = 0, \quad n < 0 \qquad \lim_{z\to\infty} X(z) = x[0]$		

Summary

- Fourier transform and frequency response
  - Frequency-domain representation of discrete-time signals and systems
  - Fourier transform theorems
- z-transform
  - Z-transform
  - Properties of the ROC
  - Inverse z-transform
  - Properties of z-transform

Course at a glance