Solutions, Exercises, MM9, Digital Signal Processing

9.1

(a)

$$\begin{aligned} x[n] &= \delta[n] \\ X[k] &= \sum_{n=0}^{N-1} \delta[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= 1 \end{aligned}$$

(b)

$$\begin{aligned} x[n] &= \delta[n-n_0], \quad 0 \le n_0 \le (N-1) \\ X[k] &= \sum_{n=0}^{N-1} \delta[n-n_0] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= W_N^{kn_0} \end{aligned}$$

(c)

$$\begin{aligned} x[n] &= \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{(N/2)-1} W_N^{2kn} \\ &= \cancel{\frac{1-e^{-j2\pi k}}{1-e^{-j(\pi k/N)}}} \qquad \frac{1-e^{-j2\pi k}}{1-e^{-j\frac{4\pi k}{N}}} \end{aligned}$$

✓ $$X[k] = \begin{cases} N/2, & k = 0, N/2 \\ 0, & \text{otherwise} \end{cases}$$

l'Hôspital!

(d)

$$x[n] = \begin{cases} 1, & 0 \le n \le ((N/2)-1) \\ 0, & N/2 \le n \le (N-1) \end{cases}$$

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\
&= \sum_{n=0}^{(N/2)-1} W_N^{kn} \\
&= \frac{1 - e^{-j\pi k}}{1 - e^{-j(2\pi k)/N}} \\
X[k] &= \begin{cases} N/2, & k = 0 \\ \dfrac{2}{1 - e^{-j(2\pi k/N)}}, & k \text{ odd} \\ 0, & k \text{ even}, 0 \le k \le (N-1) \end{cases}
\end{aligned}
$$

(e)

$$
\begin{aligned}
x[n] &= \begin{cases} a^n, & 0 \le n \le (N-1) \\ 0, & \text{otherwise} \end{cases} \\
X[k] &= \sum_{n=0}^{N-1} a^n W_N^{kn}, \quad 0 \le k \le (N-1) \\
&= \frac{1 - a^N e^{-j2\pi k}}{1 - a e^{-j(2\pi k)/N}} \\
X[k] &= \frac{1 - a^N}{1 - a e^{-j(2\pi k)/N}}
\end{aligned}
$$

9.2

Consider the finite-length sequence

$$
x[n] = \begin{cases} e^{j\omega_0 n}, & 0 \le n \le (N-1) \\ 0, & \text{otherwise} \end{cases}
$$

(a) The Fourier transform of $x[n]$:

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\
&= \sum_{n=0}^{N-1} e^{j\omega_0 n} e^{-j\omega n} \\
X(e^{j\omega}) &= \frac{1 - e^{-j(\omega-\omega_0)N}}{1 - e^{-j(\omega-\omega_0)}} \\
&= \frac{e^{-j(\omega-\omega_0)(N/2)}}{e^{-j(\omega-\omega_0)/2}} \left(\frac{\sin[(\omega-\omega_0)(N/2)]}{\sin[(\omega-\omega_0)/2]} \right) \\
X(e^{j\omega}) &= e^{-j(\omega-\omega_0)((N-1)/2)} \left(\frac{\sin[(\omega-\omega_0)(N/2)]}{\sin[(\omega-\omega_0)/2]} \right)
\end{aligned}
$$

(b) N-point DFT:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{N-1} e^{j\omega_0 n} W_N^{kn} \\ &= \frac{1 - e^{-j((2\pi k/N)-\omega_0)N}}{1 - e^{-j((2\pi k/N)-\omega_0)}} \\ &= e^{-j(\frac{2\pi k}{N}-\omega_0)(\frac{N-1}{2})} \frac{\sin\left[\left(\frac{2\pi k}{N} - \omega_0\right)\frac{N}{2}\right]}{\sin\left[\left(\frac{2\pi k}{N} - \omega_0\right)/2\right]} \end{aligned}$$

Note that $X[k] = X(e^{j\omega})|_{\omega=(2\pi k)/N}$

(c) Suppose $\omega_0 = (2\pi k_0)/N$, where k_0 is an integer:

$$\begin{aligned} X[k] &= \frac{1 - e^{-j(k-k_0)2\pi}}{1 - e^{-j(k-k_0)(2\pi)/N}} \\ &= e^{-j(2\pi/N)(k-k_0)((N-1)/2)} \frac{\sin \pi(k-k_0)}{\sin(\pi(k-k_0)/N}) \end{aligned}$$

9.3

(a)

$$x[n] = \cos(\frac{\pi n}{2}), \quad 0 \le n \le 3$$

transforms to

$$X[k] = \sum_{n=0}^{3} \cos(\frac{\pi n}{2}) W_4^{kn}, \quad 0 \le k \le 3$$

The cosine term contributes only two non-zero values to the summation, giving:

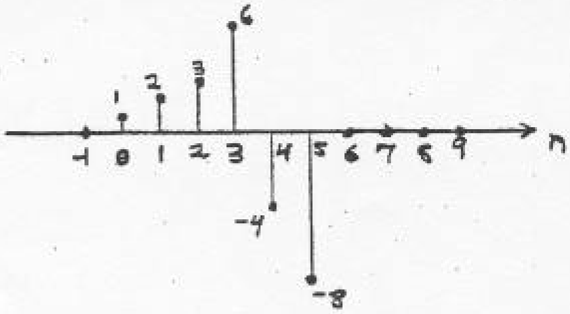
$$\begin{aligned} X[k] &= 1 - e^{-j\pi k}, \quad 0 \le k \le 3 \\ &= 1 - W_4^{2k} \end{aligned}$$

(b)

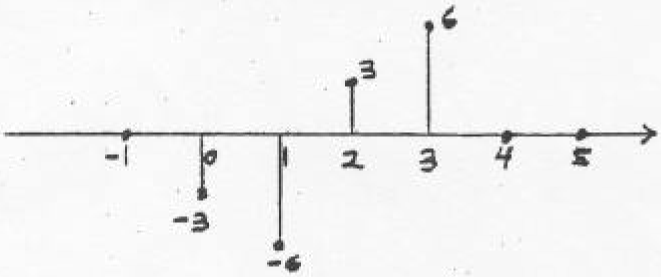
$$h[n] = 2^n, \quad 0 \le n \le 3$$

$$
\begin{aligned}
H[k] &= \sum_{n=0}^{3} 2^n W_4^{kn}, \quad 0 \le k \le 3 \\
&= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k}
\end{aligned}
$$

(c) Remember, circular convolution equals linear convolution plus aliasing. We need $N \geq 3 + 4 - 1 = 6$ to avoid aliasing. Since $N = 4$, we expect to get aliasing here. First, find $y[n] = x[n] * h[n]$:



For this problem, aliasing means the last three points ($n = 4, 5, 6$) will wrap-around on top of the first three points, giving $y[n] = x[n] \textcircled{4} h[n]$:



(d) Using the DFT values we calculated in parts (a) and (b):

$$
\begin{aligned}
Y[k] &= X[k]H[k] \\
&= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k} - W_4^{2k} - 2W_4^{3k} - 4W_4^{4k} - 8W_4^{5k}
\end{aligned}
$$

Since $W_4^{4k} = W_4^{0k}$ and $W_4^{5k} = W_4^k$

$$
Y[k] = -3 - 6W_4^k + 3W_4^{2k} + 6W_4^{3k}, \quad 0 \le k \le 3
$$

Taking the inverse DFT:

$$
y[n] = -3\delta[n] - 6\delta[n-1] + 3\delta[n-2] + 6\delta[n-3], \quad 0 \le n \le 3
$$