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Lecture 8: FIR Filter Design

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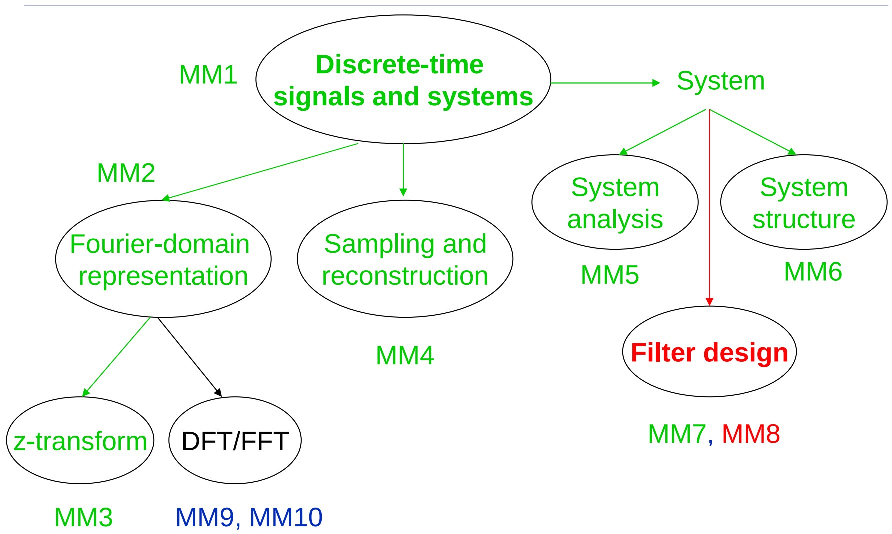
Course at a glance



FIR filter design

- Design problem: the FIR system function

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$h[n] = \begin{cases} b_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

- Start from impulse response directly

$$H(z) = h[0] + h[1]z^{-1} + ... + h[M]z^{-M}$$

Find
  - the degree M and
  - the filter coefficients $h[k]$

to approximate a desired frequency response

Part I: FIR filter design

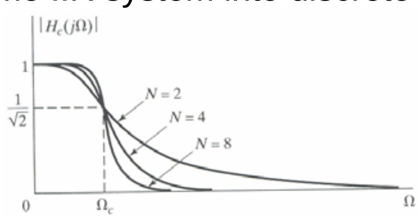
- FIR filter design
- Commonly used windows
- Generalized linear-phase FIR filter
- The Kaiser window filter design method

Lowpass filter – as an example

- Ideal lowpass filter $$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < \omega \le \pi \end{cases}$$

$$h[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- IIR filter: based on transformations of continuous-time IIR system into discrete-time ones.



$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega / \Omega_c)^{2N}}$$

poles

Figure B.2 Dependence of Butterworth magnitude characteristics on the order *N*.

- FIR filter: how? $h[n]$ is non-causal, infinite!

Design by windowing

- Desired frequency responses are often piecewise-constant with discontinuities at the boundaries between bands, resulting in non-causal and infinte impulse response extending from $-\infty$ to ∞, but

$$n \to \pm\infty, \quad h_d[n] \to 0$$

- So, the most straightforward method is to truncate the ideal response by windowing and do time-shifting:

$$g[n] = \begin{cases} h_d[n], & |n| \le M \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = g[n - M]$$

Design by windowing

After Champagne & Labeau, DSP Class Notes 2002.

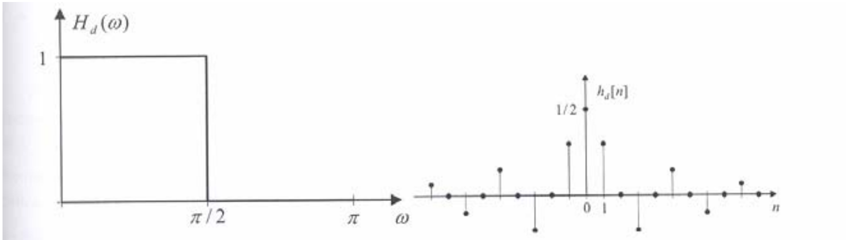


Figure 9.17: Desired (a) frequency response and (b) impulse response for example 9.7

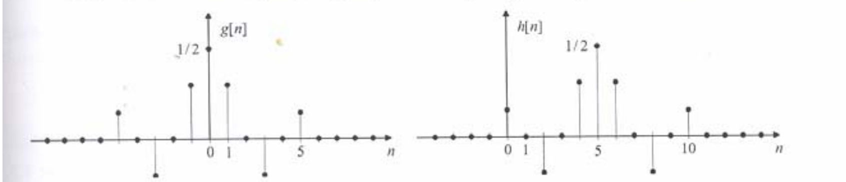


Figure 9.18: (a) Windowed impulse response $g[n]$ and (b) shifted windowed impulse response $h[n]$ for example 9.7

Design by rectangular window

- In general,

$$h[n] = h_d[n]w[n]$$

For simple truncation, the window is the rectangular window

$$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$$

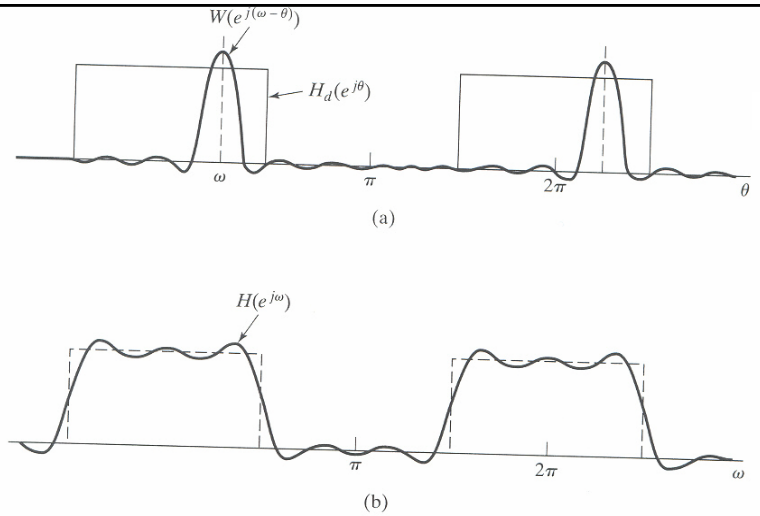


Figure 7.19 (a) Convolution process implied by truncation of the ideal impulse response. (b) Typical approximation resulting from windowing the ideal impulse response.

$w[n] = 1, \quad -\infty < n < \infty,$ then what? $\quad W(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$

$W(e^{j\omega})$ should be narrow band

Requirements on the window

$$w[n] = 1, \quad -\infty < n < \infty, \qquad W(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

$W(e^{j\omega})$ should be narrow band

- Requirements
  - $W(e^{j\omega})$ approximates an impulse to faithfully reproduce the desired frequency response
  - $w[n]$ as short as possible in duration (the order of the filter) to minimize computation in the implementation of the filter
  - → Conflicting

Take the rectangular window as an example.

Rectangular window

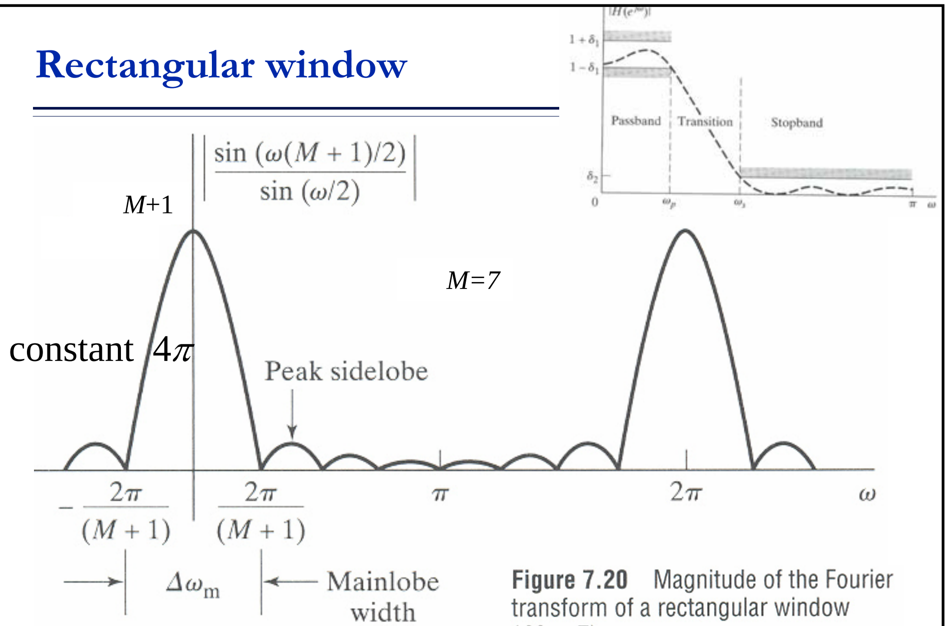


Figure 7.20 Magnitude of the Fourier transform of a rectangular window ($M = 7$).

Part II: Commonly used windows

- FIR filter design
- Commonly used windows
- Generalized linear-phase FIR filter
- The Kaiser window filter design method

Standard windows – time domain

- Rectangular

$$w[n]=\begin{cases}1, & 0\le n\le M\\ 0, & \text{otherwise}\end{cases}$$

- Triangular

$$w[n]=\begin{cases}2n/M, & 0\le n\le M/2\\ 2-2n/M, & M/2\le n\le M\\ 0, & \text{otherwise}\end{cases}$$

- Hanning

$$w[n]=\begin{cases}0.5-0.5\cos(2\pi n/M), & 0\le n\le M\\ 0, & \text{otherwise}\end{cases}$$

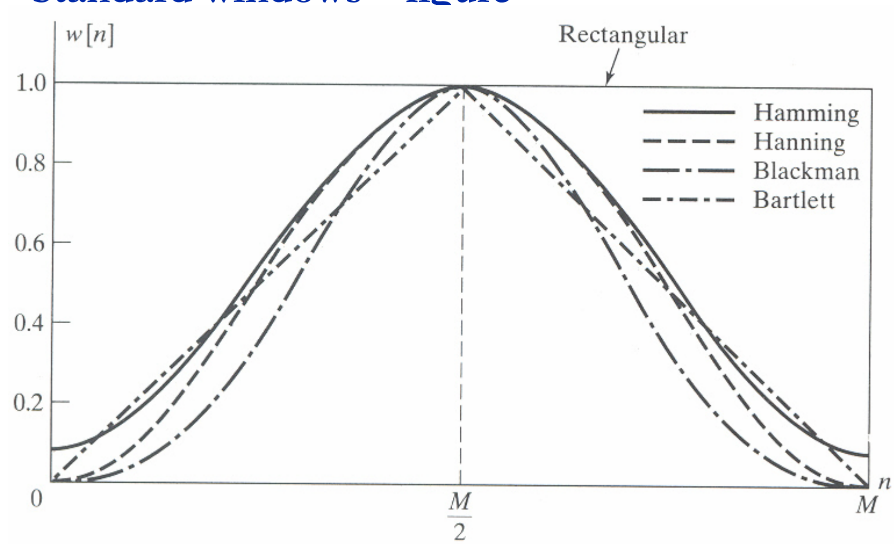
- Hamming

$$w[n]=\begin{cases}0.54-0.46\cos(2\pi n/M), & 0\le n\le M\\ 0, & \text{otherwise}\end{cases}$$

- Blackman

$$w[n]=\begin{cases}0.42-0.5\cos(2\pi n/M)+0.08\cos(4\pi n/M), & 0\le n\le M\\ 0, & \text{otherwise}\end{cases}$$

Standard windows – figure



Plotted for convenience. In fact, the window is defined only at integer values of *n*.

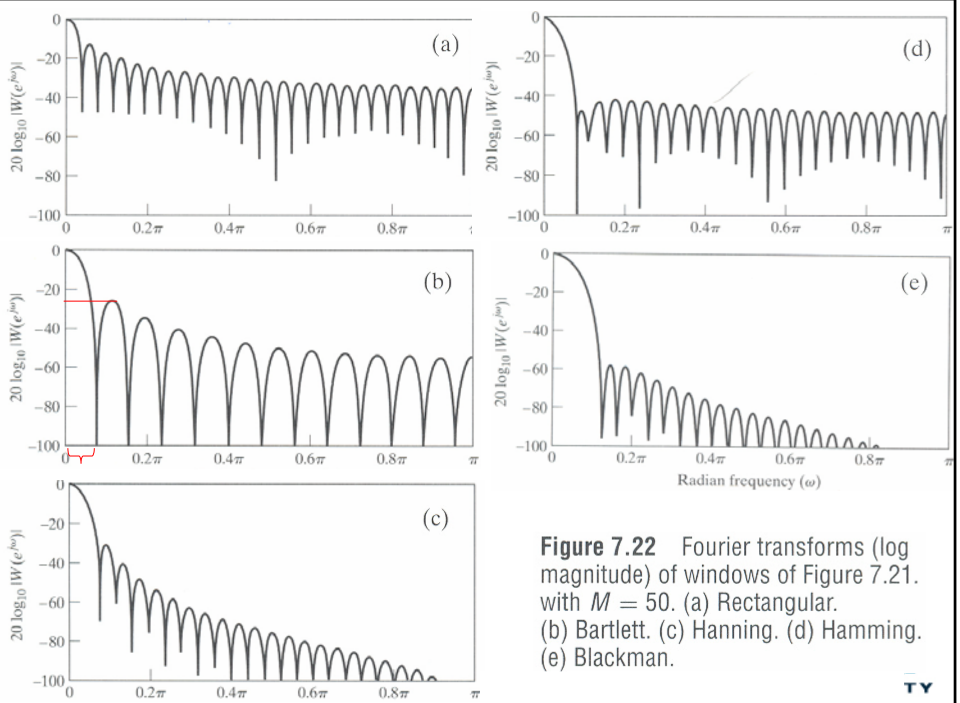


Figure 7.22 Fourier transforms (log magnitude) of windows of Figure 7.21. with $M = 50$. (a) Rectangular. (b) Bartlett. (c) Hanning. (d) Hamming. (e) Blackman.

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Standard windows – comparison

Magnitude of side lobes vs width of main lobe

TABLE 7.1 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hanning	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

Independent of M!

Part III: Linear-phase FIR filter

- FIR filter design
- Commonly used windows
- Generalized linear-phase FIR filter
- The Kaiser window filter design method

Linear-phase FIR systems

- Generalized linear-phase system

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega\alpha + j\beta}$$

$A(e^{j\omega})$ is a real function of ω,

α and β are real constants

- *Causal FIR systems* have generalized linear-phase if h[n] satisfies the symmetry condition

$$h[M-n] = h[n], \; n = 0,1,\ldots,M$$

or

$$h[M-n] = -h[n], \; n = 0,1,\ldots,M$$

Types of GLP FIR filters

$$h[M-n] = \varepsilon h[n], \ \ n = 0,1,\ldots,M$$
$$\varepsilon = 1 \text{ or } -1$$

	M even	M odd
$\varepsilon = 1$	Type I	Type II
$\varepsilon = -1$	Type III	Type IV

Generalized linear phase FIR filter

- Often aim at designing causal systems with a generalized linear phase (stability is not a problem)
  - If the impulse response of the desired filter is symmetric about M/2, $h_d[M-n] = h_d[n]$
  - Choose windows being symmetric about the point M/2

$$w[n] = \begin{cases} w[M-n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$W(e^{j\omega}) = W_e(e^{j\omega})e^{-j\omega M/2}$$

$W_e(e^{j\omega})$ is a real, even function of *w*

→ the resulting frequency response will have a generalized linear phase

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$ $A_e(e^{j\omega})$ is real and even

Linear-phase lowpass filter – an example

Desired frequency response

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega M/2}, & |\omega| < \omega_c \\ 0, & \omega_c < \omega \le \pi \end{cases}$$

$$h_{lp}[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)}, \quad -\infty < n < \infty$$

so

$$h_{lp}[M-n] = h_{lp}[n]$$

apply a symmetric window → a linear-phase system

$$h[n] = h_{lp}[n]w[n] = \frac{\sin[\omega_c(n - M/2)]}{\pi(n - M/2)} w[n]$$

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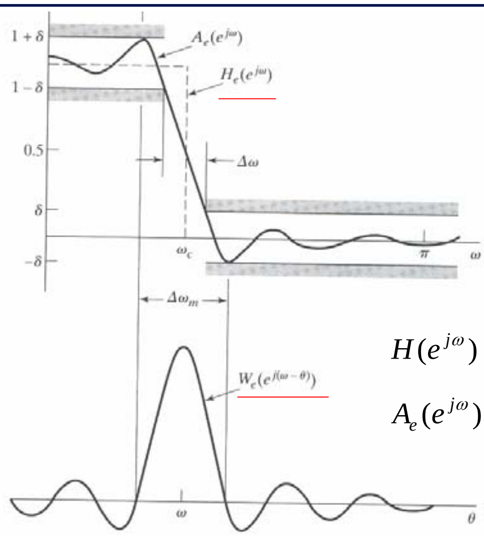
Window method approximations



$$H_d(e^{j\omega}) = H_e(e^{j\omega})e^{-j\omega M/2}$$

$H_e(e^{j\omega})$ is real and even

$$W(e^{j\omega}) = W_e(e^{j\omega})e^{-j\omega M/2}$$

$W_e(e^{j\omega})$ is real and even

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$

$$A_e(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e(e^{j\theta})W_e(e^{j(\omega-\theta)})d\theta$$

Figure 7.23 Illustration of type of approximation obtained at a discontinuity of the ideal frequency response.

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Key parameters

- To meet the requirement of FIR filter, choose
  - Shape of the window
  - Duration of the window
- Trail and error is not a satisfactory method to design filters → a simple formalization of the window method by Kaiser

Part IV: Kaiser window filter design

- FIR filter design
- Commonly used windows
- Generalized linear-phase FIR filter
- The Kaiser window filter design method

The Kaiser window filter design method

- An easy way to find the trade-off between the main-lobe width and side-lobe area
- The Kaiser window

$$w[n] = \begin{cases} \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$\alpha = M/2$$

  - $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind. $\beta \ge 0$ is an adjustable design parameter.
  - The length (M+1) and the shape parameter β can be adjusted to trade side-lobe amplitude for main-lobe width (not possible for preceding windows!)

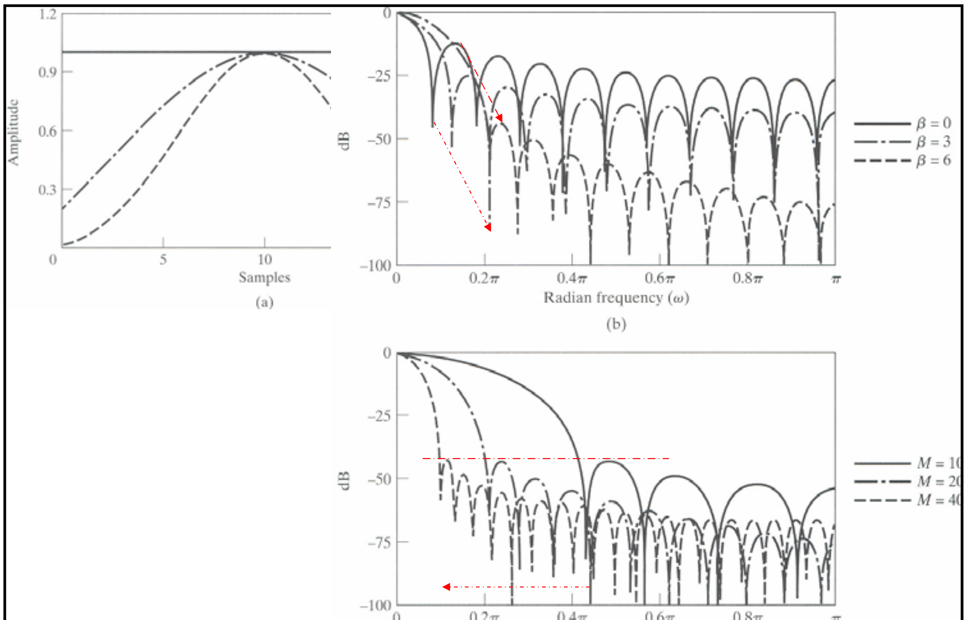


Figure 7.24 (a) Kaiser windows for $\beta = 0$, 3, and 6 and $M = 20$. (b) Fourier transforms corresponding to windows in (a). (c) Fourier transforms of Kaiser windows with $\beta = 6$ and $M = 10$, 20, and 40.

Design FIR filter by the Kaiser window

Calculate M and β to meet the filter specification

- The peak approximation error δ is determined by β

Define $A = -20\log_{10}\delta$ then *(Peak error is fixed for other windows)*

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0.0, & A < 21 \end{cases}$$ *(Rectangular)*

- Passband cutoff frequency ω_p is determined by:

$|H(e^{j\omega})| \ge 1-\delta$

Stopband cutoff frequency ω_s by: $|H(e^{j\omega})| \le \delta$

Transition width $\Delta\omega = \omega_s - \omega_p$

M must satisfy $M = \dfrac{A-8}{2.285\Delta\omega}$

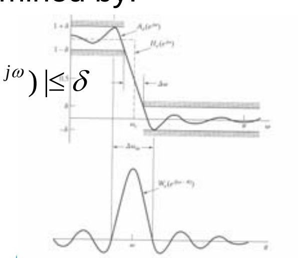


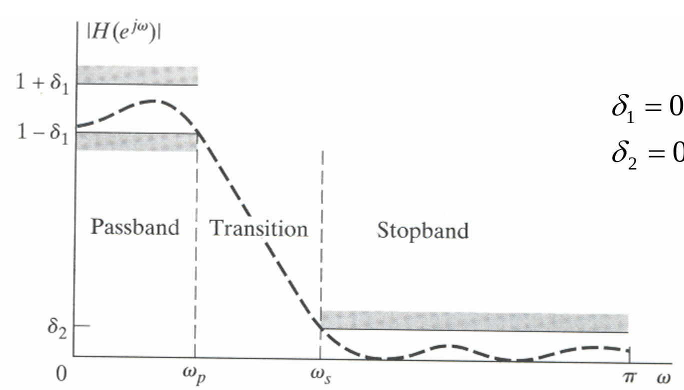
Figure 7.23 Illustration of type of approximation obtained at a discontinuity of the ideal frequency response.

A lowpass filter Specifications

- Specifications for a discrete-time lowpass filter

$$1-0.01 \le |H(e^{j\omega})| \le 1+0.01, \quad 0 \le \omega \le \omega_p = 0.4\pi$$

$$|H(e^{j\omega})| \le 0.001, \quad \omega \ge \omega_s = 0.6\pi$$



$\delta_1 = 0.01$

$\delta_2 = 0.001$

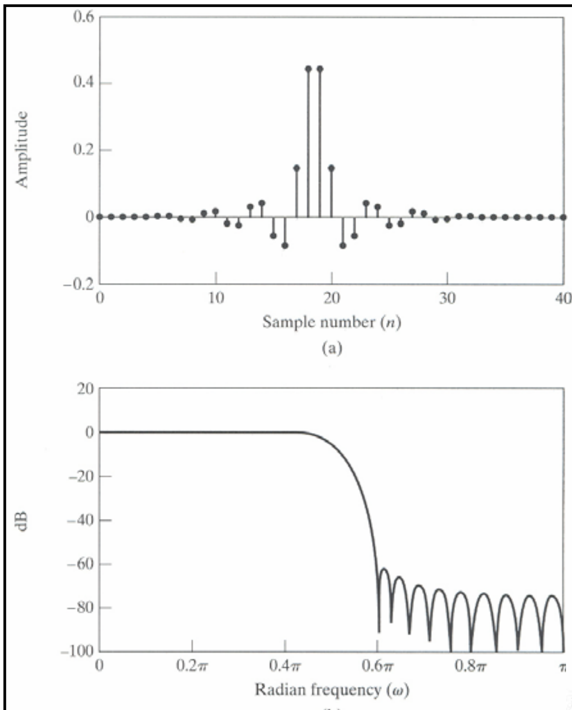
Design the lowpass filter by Kaiser window

- Designing by window method indicating $\delta_1 = \delta_2$, we must set $\delta = 0.001$
- Transition width $\Delta\omega = \omega_s - \omega_p = 0.2\pi$
- $A = -20\log_{10}\delta = 60$
- The two parameters: $\beta = 5.653, \quad M = 37$
- Cutoff frequency of the ideal lowpass filter

$$\omega_c = (\omega_s + \omega_p)/2 = 0.5\pi$$

- Impulse response

$$h[n] = \begin{cases} \dfrac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} \cdot \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{1/2}]}{I_0(\beta)}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$



What is the group delay?

$M/2 = 18.5$

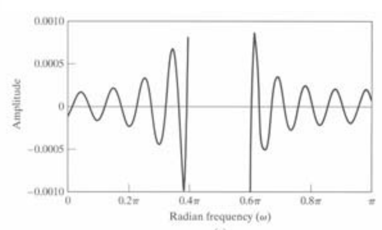


Figure 7.25 Response functions for Example 7.8. (a) Impulse response ($M = 37$). (b) Log magnitude. (c) Approximation error.

Summary

- FIR filter design
- Commonly used windows
- Generalized linear-phase FIR filter
- The Kaiser window filter design method

Course at a glance