Digital Signal Processing, Fall 2006

Lecture 7: Filter Design

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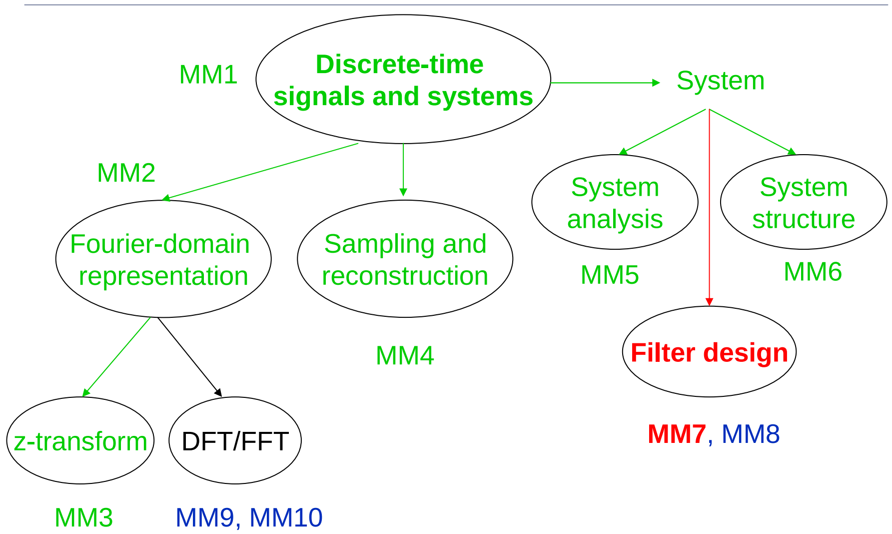
Course at a glance



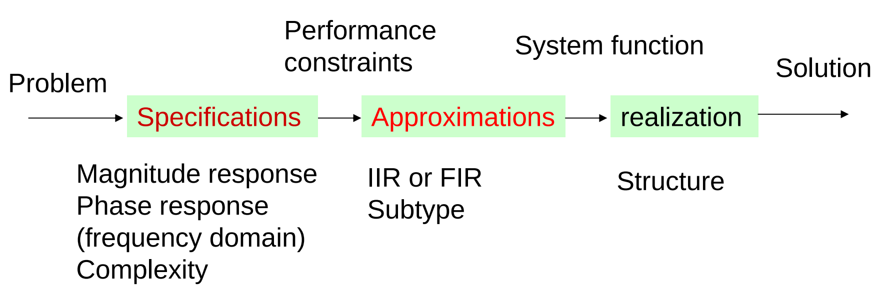
Part I: Filter design

- Filter design
- IIR filter design
- Analog filter design
- IIR filter design by impulse invariance
- IIR filter design by bilinear transformation

Filter design process

Filter, in broader sense, covers any system.

Three design steps

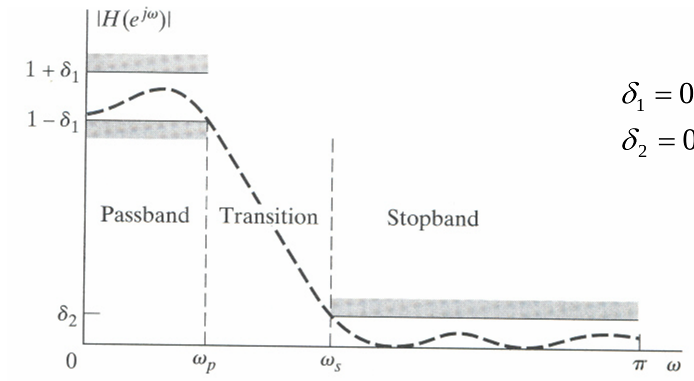


Specifications – an example

- Specifications for a discrete-time lowpass filter

$$1-0.01 \le |H(e^{j\omega})| \le 1+0.01, \quad 0 \le \omega \le \omega_p$$

$$|H(e^{j\omega})| \le 0.001, \quad \omega \ge \omega_s$$



$$\delta_1 = 0.01$$

$$\delta_2 = 0.001$$

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Specifications of frequency response

- Typical lowpass filter specifications in terms of tolerable
  - Passband distortion, as smallest as possible
  - Stopband attenuation, as greatest as possible
  - Width of transition band: as narrowest as possible

- Improving one often worsens others → a tradeoff
- Increasing filter order improves all

DT filter for CT signals

- Discrete-time filter for the processing of continuous-time signals
  - Bandlimited input signal
  - High enough sampling frequency
- Then, specifications conversion is straightforward

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi / T \\ 0, & |\Omega| > \pi / T \end{cases}$$

$$H(e^{j\omega}) = H_{eff}(j\frac{\omega}{T}), \qquad |\omega| < \pi \qquad\qquad \omega = \Omega T$$

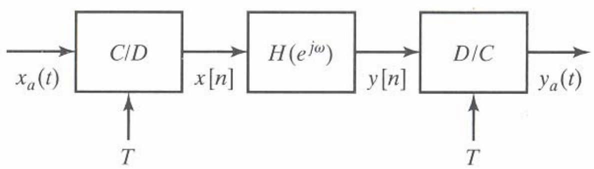


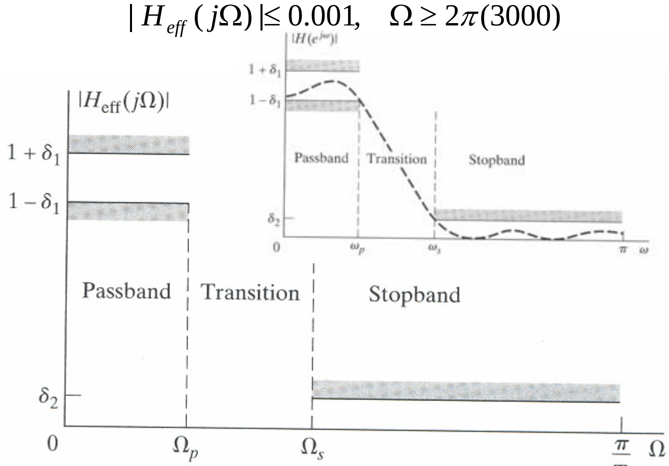
Figure 7.1 Basic system for discrete-time filtering of continuous-time signals.

Specifications – an example

- Specifications for a continuous-time lowpass filter

$$1 - 0.01 \le |H_{eff}(j\Omega)| \le 1 + 0.01, \quad 0 \le \Omega \le 2\pi(2000)$$

$$|H_{eff}(j\Omega)| \le 0.001, \quad \Omega \ge 2\pi(3000)$$



$\delta_1 = 0.01$

$\delta_2 = 0.001$

$\Omega_p = 2\pi(2000)$

$\Omega_s = 2\pi(3000)$

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Design a filter

- Design goal: find system function to make frequency response meet the specifications (tolerances)
- Infinite impulse response filter
  - Poles insider unit circle due to causality and stability
  - Rational function approximation
- Finite impulse response filter
  - Linear phase is often required
  - Polynomial approximation

E.g. IIR filter design

- For rational system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

find the system coefficients such that the corresponding frequency response

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$$

provides a good approximation to a desired response

$$H(e^{j\omega}) \approx H_{desired}(e^{j\omega}) \longrightarrow$$

H(z)
•Rational system function
•Stable
•causal

FIR or IIR

- Either FIR or IIR is often dependent on the phase requirements
- Only FIR filter can be at the same time stable, causal and GLP
  - If $H(z)$ is stable and GLP, any non-trivial pole p inside the unit circle corresponds a pole $1/p$ outside the unit circle, so that H(z) cannot have a causal impulse response (as ROC is a ring including unit circle).
- Design principle
  - If GLP is essential → FIR
  - If not → IIR preferable (can meet specifications with lower complexity)

FIR and IIR

IIR
- Rational system function
- Poles + zeros
- Stable/unstable
- Hard to control phase
- Low order (4-20)
- Designed on the basis of analog filter

FIR
- Polynomial system function
- Zeros
- Stable
- Easy to get linear phase
- High order (20-2000)
- Unrelated to analog filter

Part II: IIR filter design

- Filter design
- IIR filter design
- Analog filter design
- IIR filter design by impulse invariance
- IIR filter design by bilinear transformation

Design IIR filter based on analog filter

- The mapping is direct

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi / T \\ 0, & |\Omega| > \pi / T \end{cases}$$

$$H(e^{j\omega}) = H_{eff}(j\frac{\omega}{T}), \qquad |\omega| < \pi$$

- Advanced analog filter design techniques

→ Designing DT filter by transforming prototype CT filter:
  - Transform (map) DT specifications to analog
  - Design analog filter
  - Inverse-transform analog filter to DT

Transformation method

- Transform (map) DT specifications to analog

$$\omega = \Omega T$$

- Design analog filter

$$H_c(s) \text{ or } h_c(t)$$

- Inverse-transform to DT

$$H(z) \text{ or } h[n]$$

• The imaginary axis of the s-plane → the unit circle of the z-plane
• Poles in the left half of the s-plane → poles inside the unit circle in the z-plane (stable)

$$s = \sigma + j\Omega$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

$$H(j\Omega) = \int_{-\infty}^{\infty} h(t)e^{-j\Omega t}dt$$

$$z = re^{-j\omega}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Part III: Analog filter design

- Filter design
- IIR filter design
- Analog filter design
- IIR filter design by impulse invariance
- IIR filter design by bilinear transformation

Analog filter design

- Butterworth
- Chebyshev I
- Chebyshev II
- Ellipical

Butterworth lowpass filters

- The magnitude response
  - Maximally flat in the passband
  - Monotonic in both passband and stopband
- The squared magnitude response

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

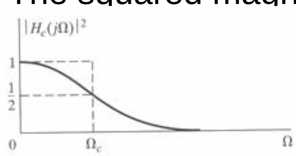


Figure B.1 Magnitude-squared function for continuous-time Butterworth filter.

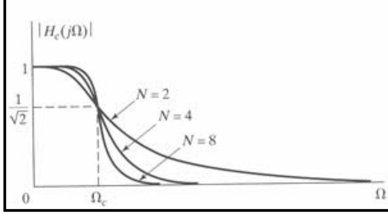


Figure B.2 Dependence of Butterworth magnitude characteristics on the order N.

Poles in s-plane

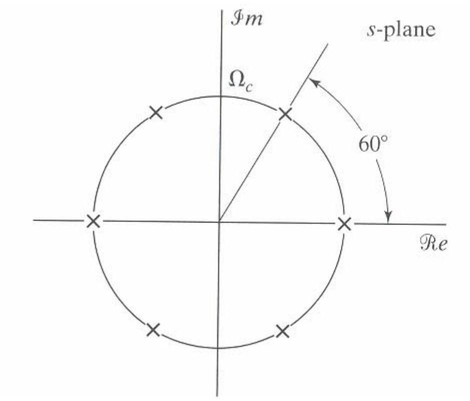


Figure B.3 s-plane pole locations for a third-order Butterworth filter

Chebyshev filters

- Chebyshev II filters: equiripple in passband, flat in the stopband

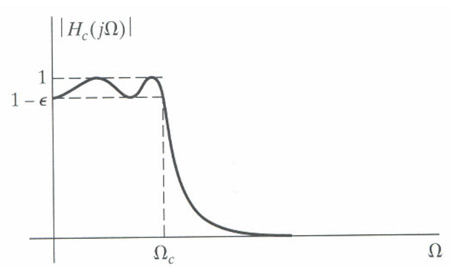


Figure B.4 Type I Chebyshev lowpass filter approximation.

- Chebyshev II filters: equiripple in stopband, flat in the passband

Elliptic filters

- Equiripple both in stopband and in the passband

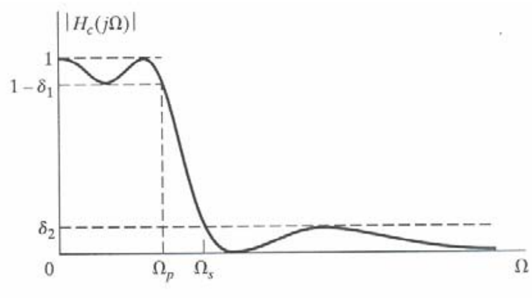


Figure B.6 Equiripple approximation in both passband and stopband.

Part IV: Design by impulse invariance

- Filter design
- IIR filter design
- Analog filter design
- IIR filter design by impulse invariance
- IIR filter design by bilinear transformation

Filter design by impulse invariance

- Impulse invariance: a method for obtaining a DT system whose $H(e^{j\omega})$ is determined by the $H_c(j\Omega)$ of a CT system.

$$h[n] = T_d h_c(nT_d)$$

$$T_d \text{ - 'design' sampling interval}$$

  - In DT filter design, the specifications are provided in the discrete-time, so T_d has no role. T_d is included for discussion though. T_d also has nothing to do with C/D and D/C conversion in Fig. 7.1

Relationship btw frequency responses

- Impulse response

$$h[n] = T_d h_c(nT_d)$$

- Frequency response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k)$$

sampling:

$$x[n] = x_c(nT)$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j\frac{\omega}{T} - j\frac{2\pi k}{T})$$

if the CT filter is bandlimited

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T_d$$

then

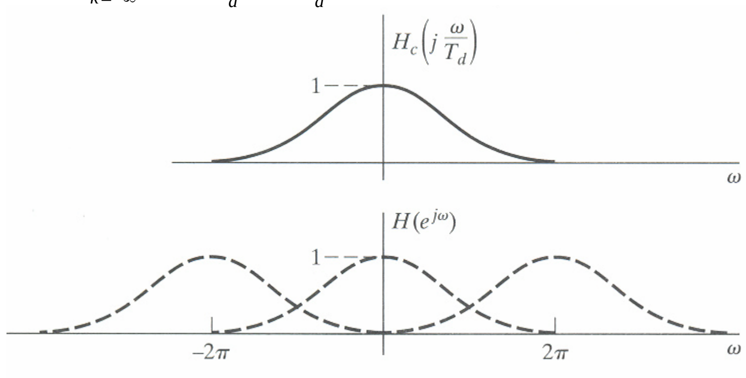
$$H(e^{j\omega}) = H_c(j\frac{\omega}{T_d}), \quad |\omega| \leq \pi$$

This is also the way to get CT filter specifications from $H(e^{j\omega})$ by applying the relation $\Omega = \omega / T_d$

Aliasing in the impulse invariance design

$$h[n] = T_d h_c(nT_d)$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi}{T_d}k\right)$$



Relationship btw system functions

- The transform from CT to DT is easy to carry out as a transformation on the system function
- Rational system function, after partial fraction expansion

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

$$h_c(t) = \begin{cases} \sum_{k=1}^{N} A_k e^{s_k t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

$$h[n] = T_d h_c(nT_d) = \sum_{k=1}^{N} T_d A_k e^{s_k n T_d} u[n] = \sum_{k=1}^{N} T_d A_k (e^{s_k T_d})^n u[n]$$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

Impulse invariance with a Butterworth filter

- Specifications

$$0.89125 \le |H(e^{j\omega})| \le 1, \quad 0 \le |\omega| \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.17783, \quad 0.3\pi \le |\omega| \le \pi$$

- Since the sampling interval Td cancels in the impulse invariance procedure, we choose Td=1, so $\omega = \Omega$
- Magnitude function for a CT Butterworth filter

$$0.89125 \le |H_c(j\Omega)| \le 1, \quad 0 \le |\Omega| \le 0.2\pi$$

$$|H_c(j\Omega)| \le 0.17783, \quad 0.3\pi \le |\Omega| \le \pi$$

- Due to the monotonic function of Butterworth filter

$$|H_c(j0.2\pi)| \ge 0.89125$$

$$|H_c(j0.3\pi)| \le 0.17783$$

Impulse invariance with a Butterworth filter

- Squared magnitude function of a Butterworth filter

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

(1) ← $|H_c(j0.2\pi)| \ge 0.89125$, $|H_c(j0.3\pi)| \le 0.17783$

(2) ↓

$$1+\left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2$$

$$1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

(3) ↓

$$N = 5.8858$$

$$\Omega_c = 0.70474$$

(4) $N = 6$

(5) $\Omega_c = 0.7032$

↓

$$H_c(s)H_c(-s) = \frac{1}{1+(s/j\Omega_c)^{2N}} = \frac{1}{1+(s/j0.7032)^{12}}$$

Impulse invariance with a Butterworth filter

- 12 poles for the squared magnitude function
- The system function has the three pole pairs in the left half of the s-plane

Pole pair 1: $-0.182 \pm j(0.679)$,

Pole pair 2: $-0.497 \pm j(0.497)$,

Pole pair 3: $-0.679 \pm j(0.182)$.

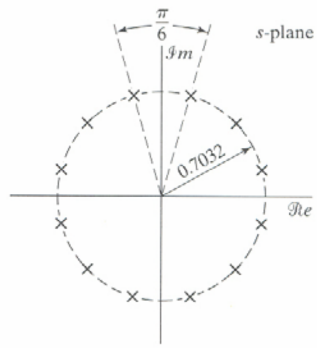


Figure 7.4 s-plane locations for poles of $H_c(s)H_c(-s)$ for sixth-order Butterworth filter in Example 7.2.

Impulse invariance with a Butterworth filter

$$H_c(s) = \frac{0.12093}{(s^2 + 0.3640s + 0.4945)(s^2 + 0.9945s + 0.4945)(s^2 + 1.3585s + 0.4945)}$$

$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{(1 - 1.2971z^{-1} + 0.6949z^{-2})}$$

$$+ \frac{-2.1428 + 1.1455z^{-1}}{(1 - 1.0691z^{-1} + 0.3699z^{-2})} + \frac{1.8557 - 0.6303z^{-1}}{(1 - 0.9972z^{-1} + 0.2570z^{-2})}$$

Impulse invariance with a Butterworth filter

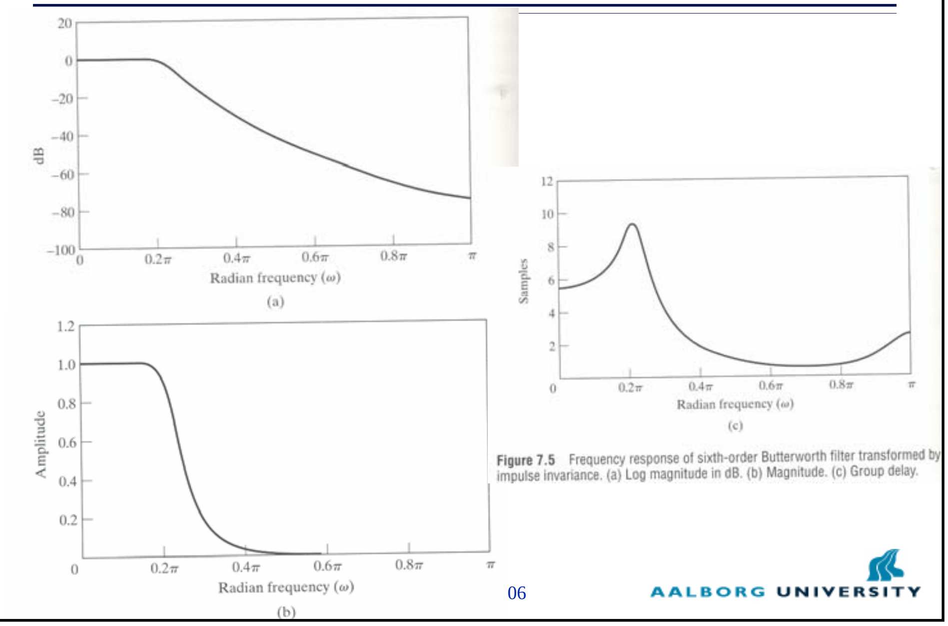


Figure 7.5 Frequency response of sixth-order Butterworth filter transformed by impulse invariance. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

Part V: Design by bilinear transformation

- Filter design
- IIR filter design
- Analog filter design
- IIR filter design by impulse invariance
- IIR filter design by bilinear transformation

Bilinear transformation

- By using impulse invariance, the relation between CT and DT frequency is linear (except for aliasing), thus the shape of the frequency response is preserved. But only proper for bandlimited filters, problem for e.g. highpass

- Bilinear transformation between s and z

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]$$

- Inverse

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s}$$

Bilinear transformation

Given $s = \sigma + j\Omega$

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s} = \frac{1+\sigma T_d/2 + j\Omega T_d/2}{1-\sigma T_d/2 - j\Omega T_d/2}$$

if $\sigma < 0$, $|z| < 1$ for any Ω

if $\sigma > 0$, $|z| > 1$ for any Ω

if $s = j\Omega$

$$z = \frac{1+j\Omega T_d/2}{1-j\Omega T_d/2}$$

so, $|z| = 1$, for any s on the $j\Omega$-axis

i.e. the $j\Omega$-axis maps onto the unit circle

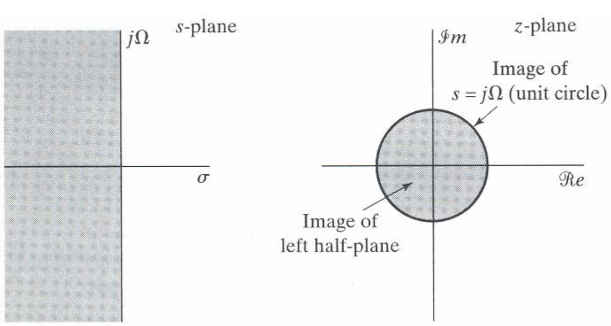


Figure 7.6 Mapping of the s-plane onto the z-plane using the bilinear transformation.

Bilinear transformation – frequency relationship

Consider frequency

$$s = \frac{2}{\ } = e^{j\omega}\ s = j\Omega\ \frac{1+j\Omega T_d}{1-j\Omega T_d}$$

$$s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$j\Omega = \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right) = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2}(j\sin\omega/2)}{2e^{-j\omega/2}(\cos\omega/2)}\right] = \frac{2j}{T_d}\tan(\omega/2)$$

$$\Omega = \frac{2}{T_d}\tan(\omega/2)$$

$$\omega = 2\arctan(\Omega T_d/2)$$

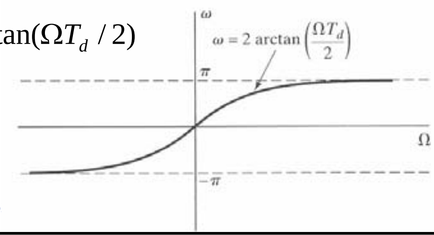


Figure 7.7 Mapping of the continuous-time frequency axis onto the discrete-time frequency axis by bilinear transformation.

Bilinear transformation

- The bilinear transformation maps the entire $j\Omega$-axis in the s-plane to one revolution of the unit circle in the z-plane.

Compare with $\Omega = \omega / T_d$

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Bilinear transformation of a Butterworth filter

- Specifications

$$0.89125 \le |H(e^{j\omega})| \le 1, \quad 0 \le |\omega| \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.17783, \quad 0.3\pi \le |\omega| \le \pi$$

- Magnitude function for a CT Butterworth filter

$$0.89125 \le |H_c(j\Omega)| \le 1, \quad 0 \le |\Omega| \le \frac{2}{T_d}\tan(\frac{0.2\pi}{2})$$
$$|H_c(j\Omega)| \le 0.17783, \quad \frac{2}{T_d}\tan(\frac{0.3\pi}{2}) \le |\Omega| \le \infty$$

- Due to the monotonic function of Butterworth filter

Choose $T_d = 1$ $\quad |H_c(j2\tan(0.1\pi))| \ge 0.89125$

$$|H_c(j2\tan(0.15\pi))| \le 0.17783$$

Bilinear transformation of a Butterworth filter

- Squared magnitude function of a Butterworth filter

$$|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \longleftarrow \begin{array}{l} |H_c(j2\tan(0.1\pi))| \ge 0.89125 \\ |H_c(j2\tan(0.15\pi))| \le 0.17783 \end{array}$$

$\downarrow$

$$N = 5.305$$

$\downarrow$

$$N = 6$$
$$\Omega_c = 0.766$$

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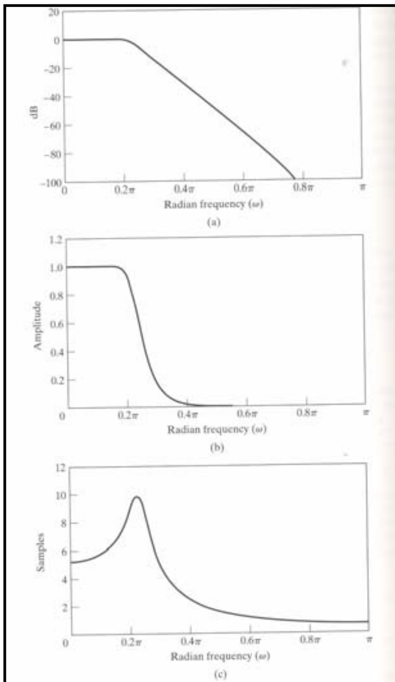

Figure 7.11 Frequency response of sixth-order Butterworth filter transformed by bilinear transform. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

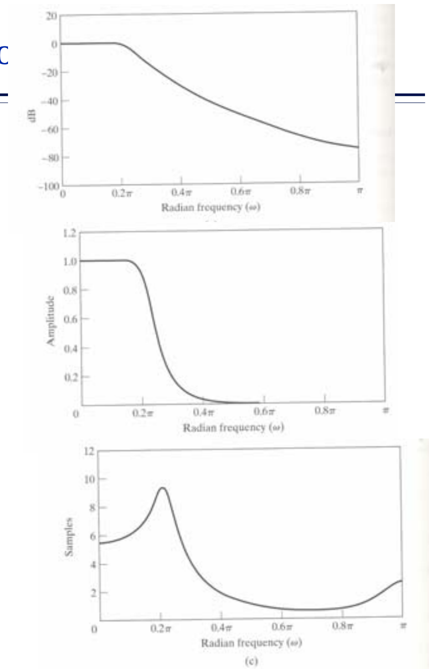

Figure 7.5 Frequency response of sixth-order Butterworth filter transformed by impulse invariance. (a) Log magnitude in dB. (b) Magnitude. (c) Group delay.

Summary

- Filter design
- IIR filter design
- Analog filter design
- IIR filter design by impulse invariance
- IIR filter design by bilinear transformation

Course at a glance