Digital Signal Processing, Fall 2006

Lecture 6: System structures for implementation

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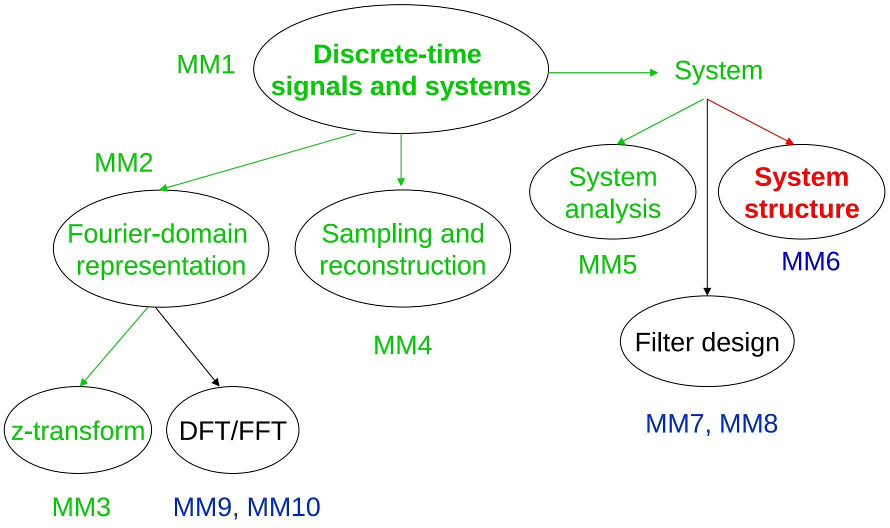
Course at a glance



MM1 **Discrete-time signals and systems**

System

MM2 Fourier-domain representation

Sampling and reconstruction

MM4

System analysis

MM5

System structure

MM6

Filter design

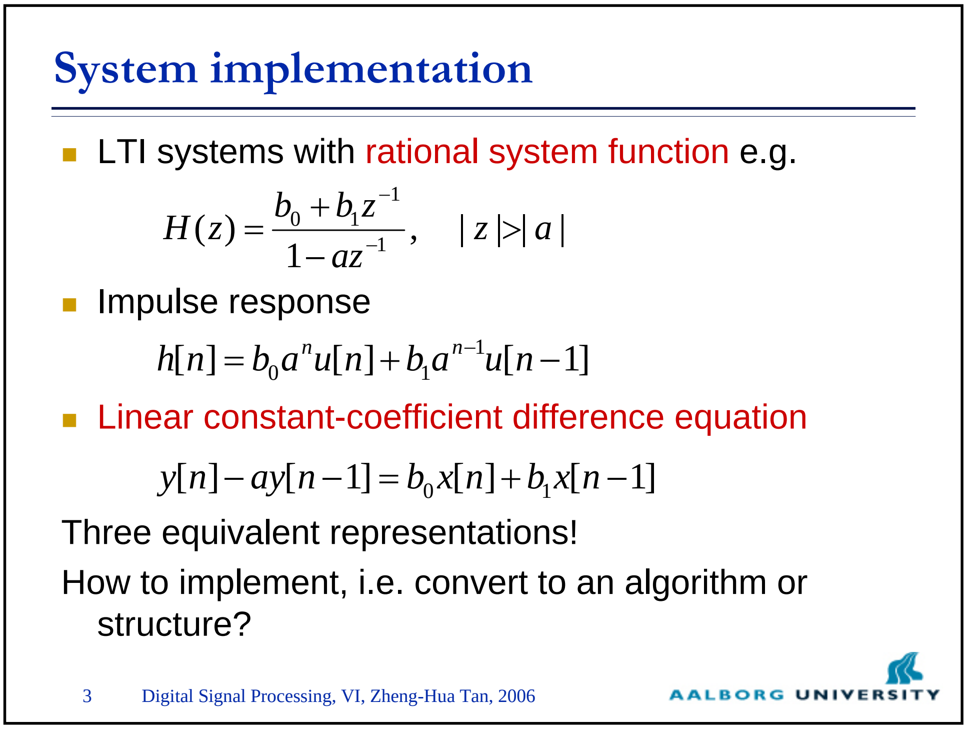
MM7, MM8

z-transform

MM3

DFT/FFT

MM9, MM10



System implementation

- LTI systems with rational system function e.g.

$$H(z)=\frac{b_0+b_1z^{-1}}{1-az^{-1}}, \quad |z|>|a|$$

- Impulse response

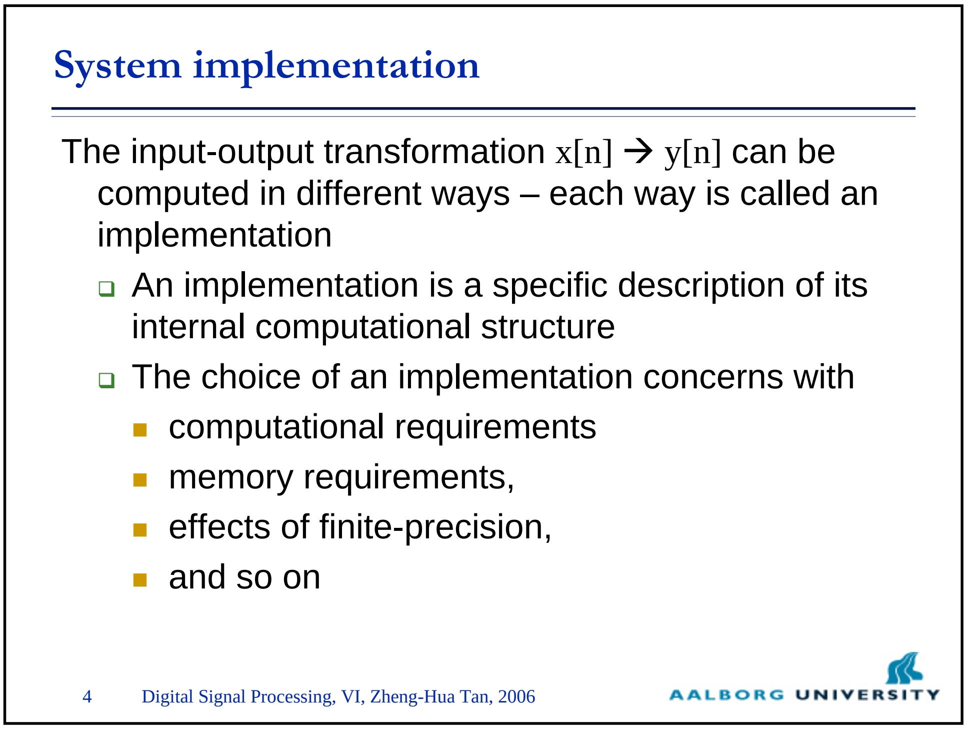
$$h[n]=b_0a^nu[n]+b_1a^{n-1}u[n-1]$$

- Linear constant-coefficient difference equation

$$y[n]-ay[n-1]=b_0x[n]+b_1x[n-1]$$

Three equivalent representations!

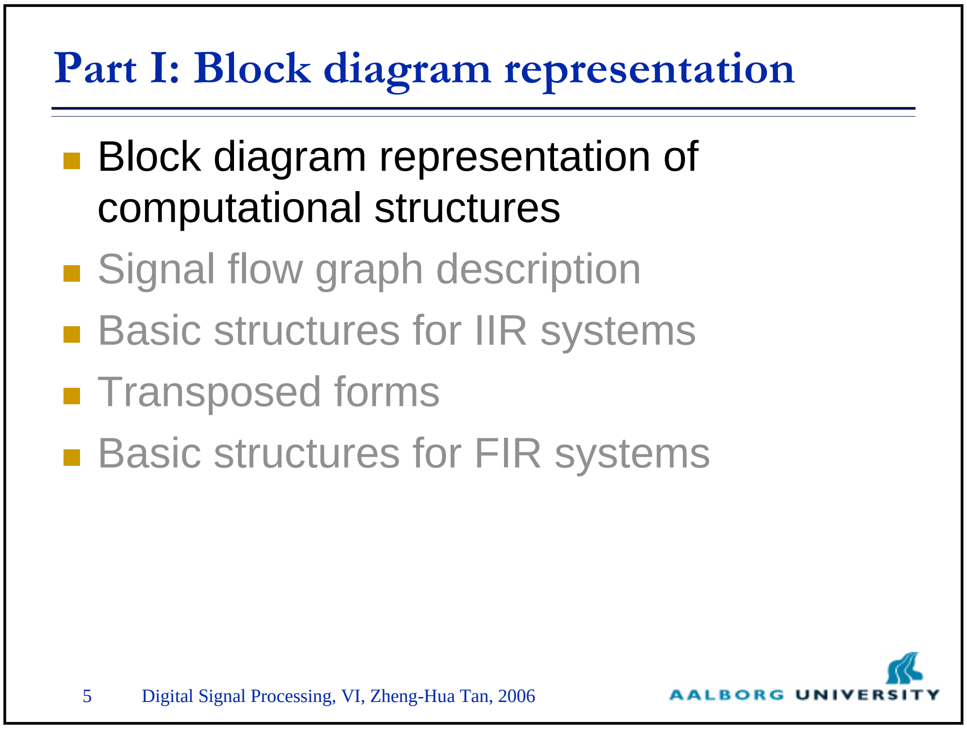
How to implement, i.e. convert to an algorithm or structure?



System implementation

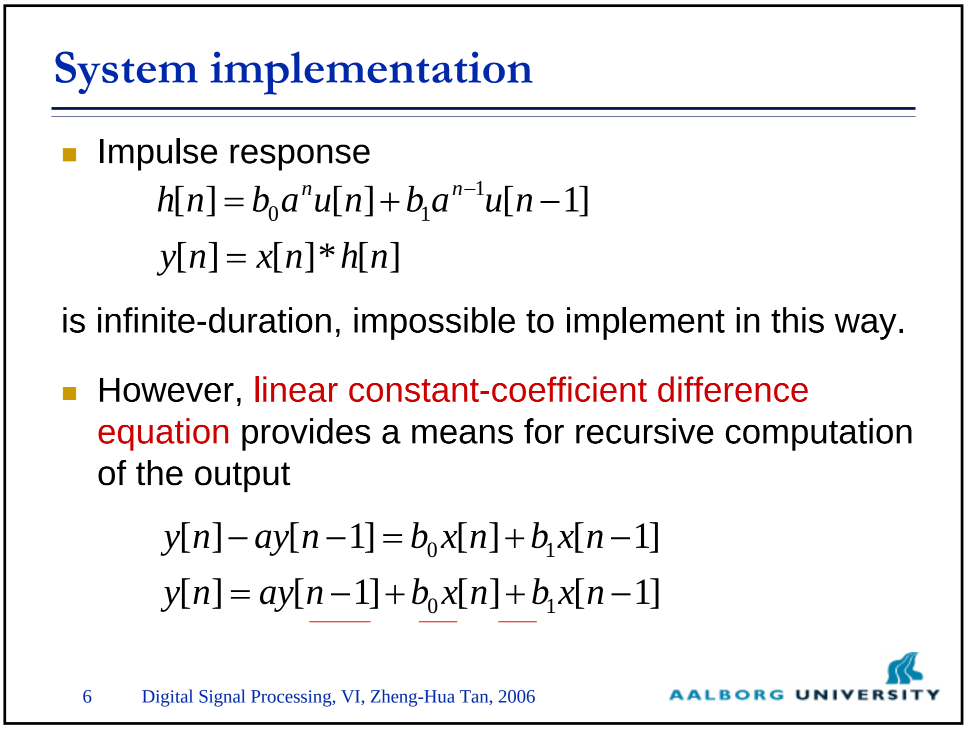
The input-output transformation x[n] → y[n] can be computed in different ways – each way is called an implementation

- An implementation is a specific description of its internal computational structure
- The choice of an implementation concerns with
  - computational requirements
  - memory requirements,
  - effects of finite-precision,
  - and so on



Part I: Block diagram representation

- Block diagram representation of computational structures
- Signal flow graph description
- Basic structures for IIR systems
- Transposed forms
- Basic structures for FIR systems



System implementation

- Impulse response

$$h[n] = b_0 a^n u[n] + b_1 a^{n-1} u[n-1]$$

$$y[n] = x[n] * h[n]$$

is infinite-duration, impossible to implement in this way.

- However, linear constant-coefficient difference equation provides a means for recursive computation of the output

$$y[n] - ay[n-1] = b_0 x[n] + b_1 x[n-1]$$

$$y[n] = ay[\underline{n-1}] + \underline{b_0} x[n] + \underline{b_1} x[n-1]$$

Basic elements

- Implementation based on the recurrence formula derived from difference equation requires
  - adders
  - multipliers
  - memory for storing delayed sequence values

$$y[n] = ay[n-1] + b_0x[n] + b_1x[n-1]$$

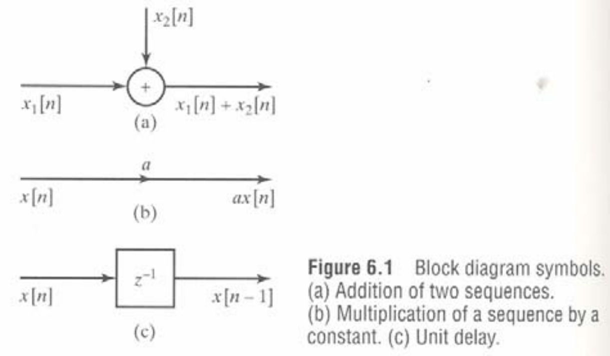


Figure 6.1 Block diagram symbols. (a) Addition of two sequences. (b) Multiplication of a sequence by a constant. (c) Unit delay.

Example of block diagram representation

- A second-order difference equation

$$y[n] = a_1y[n-1] + a_2y[n-2] + b_0x[n]$$

$$H(z) = \frac{b_0}{1 - a_1z^{-1} - a_2z^{-2}}$$

Demonstrates the complexity, the steps, the amount of resources required.

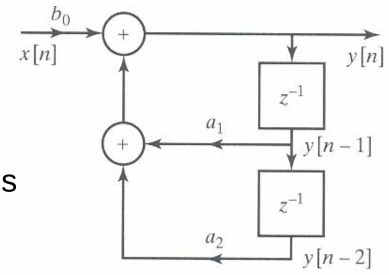
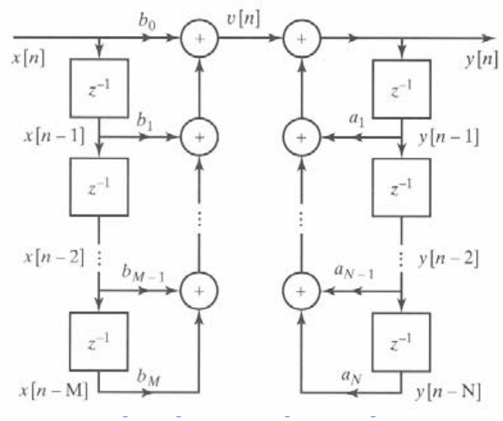


Figure 6.2 Example of a block diagram representation of a difference equation.

General Nth-order difference equation

$$y[n]=\sum_{k=1}^{N}a_k y[n-k]+\underbrace{\sum_{k=0}^{M}b_k x[n-k]}_{v[n]}$$

$$H(z)=\frac{\sum_{k=0}^{M}b_k z^{-k}}{1-\sum_{k=1}^{N}a_k z^{-k}}$$



A cascade of two systems!

X[n]→v[n], v[n]→y[n]

Figure 6.3 Block diagram representation for a general *N*th-order difference equation.

Rearrangement of block diagram

- A block diagram can be rearranged in many ways without changing overal function, e.g. by reversing the order of the two cascaded systems.

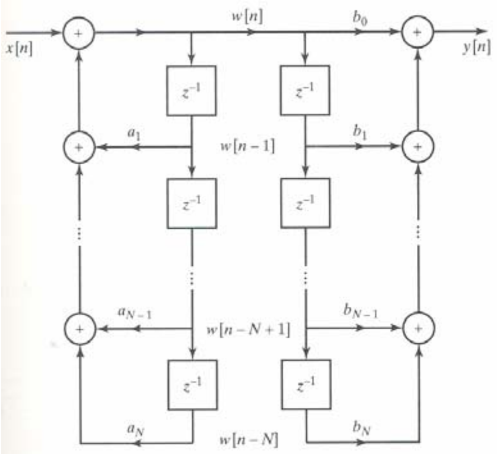
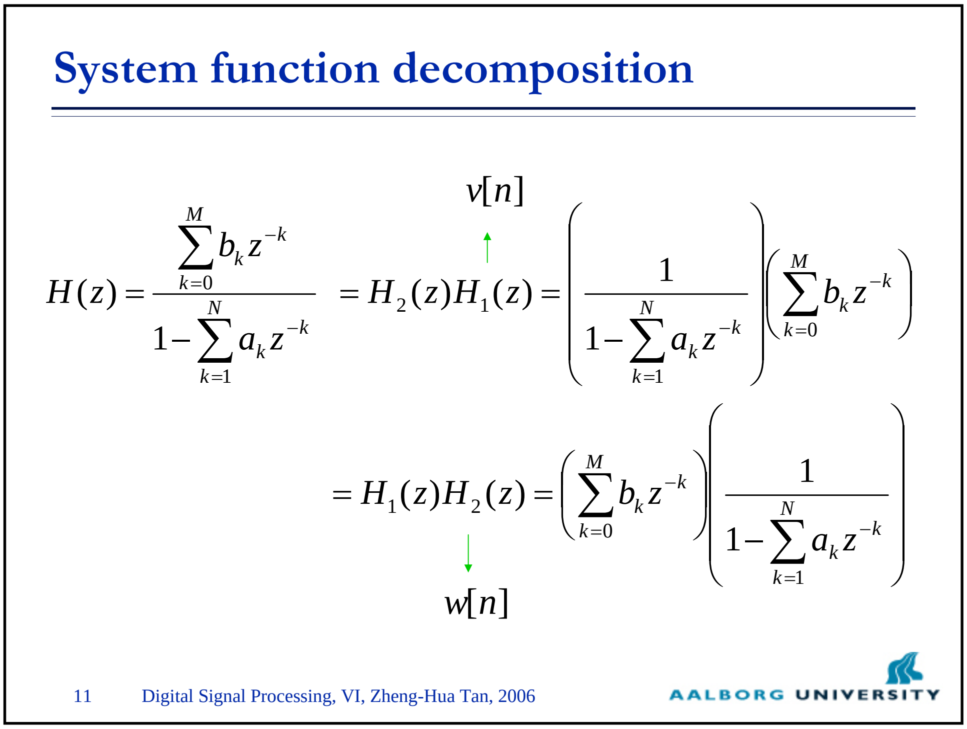


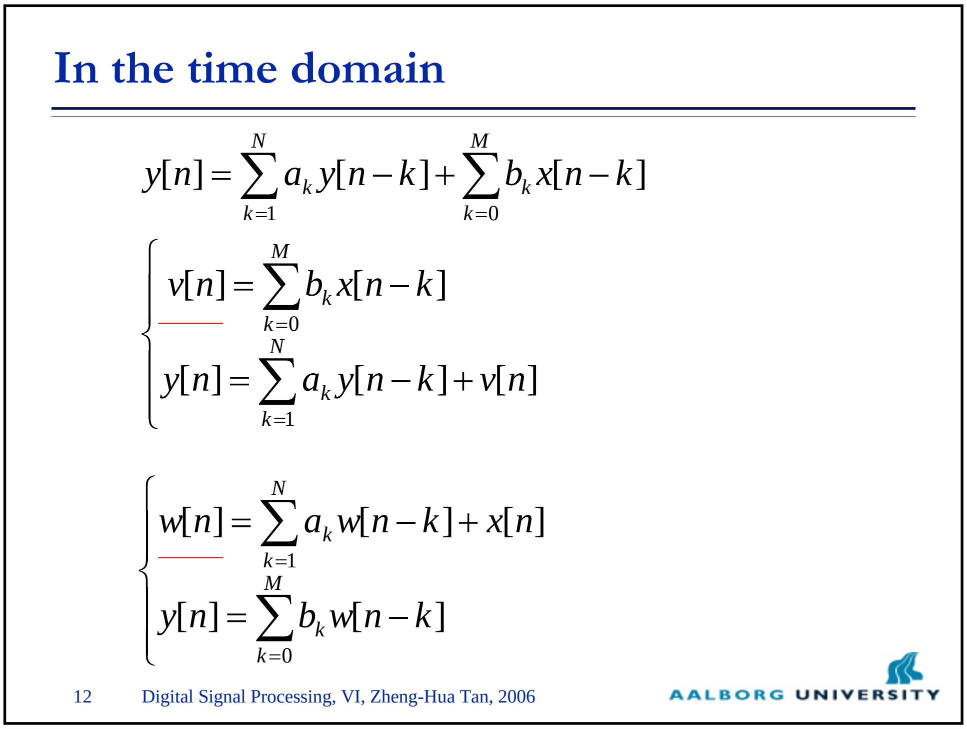
Figure 6.4 Rearrangement of block diagram of Figure 6.3. We assume for convenience that $N = M$. If $N \neq M$, some of the coefficients will be zero.



System function decomposition

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}} = \overset{\overset{\displaystyle v[n]}{\uparrow}}{H_2(z)H_1(z)} = \left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

$$= \underset{\underset{\displaystyle w[n]}{}}{\underset{\downarrow}{H_1(z)H_2(z)}} = \left(\sum_{k=0}^{M} b_k z^{-k}\right)\left(\frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}\right)$$



In the time domain

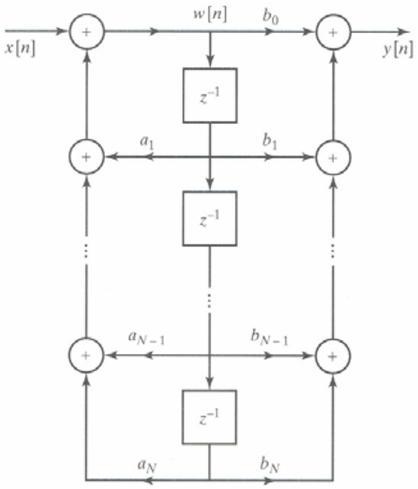
$$y[n]=\sum_{k=1}^{N} a_k y[n-k]+\sum_{k=0}^{M} b_k x[n-k]$$

$$\begin{cases} \underline{v[n]}=\sum_{k=0}^{M} b_k x[n-k] \\ y[n]=\sum_{k=1}^{N} a_k y[n-k]+v[n] \end{cases}$$

$$\begin{cases} \underline{w[n]}=\sum_{k=1}^{N} a_k w[n-k]+x[n] \\ y[n]=\sum_{k=0}^{M} b_k w[n-k] \end{cases}$$

Minimum delay implementation

- One big difference btw the two implementations concerns the number of delay elements



$$N+M$$

$$\max(N,M)$$

Figure 6.5 Combination of delays in Figure 6.4.

Direct form I and II

- Direct form I as shown in Fig. 6.3
  - A direct realization of the difference equation
- Direct form II or canonic direct form as shown in Fig. 6.5
  - There is a direct link between the system function (difference equation) and the block diagram

An example

- Direct form I and direct form II implementation

$$H(z)=\frac{1+2z^{-1}}{1-1.5z^{-1}+0.9z^{-2}}$$

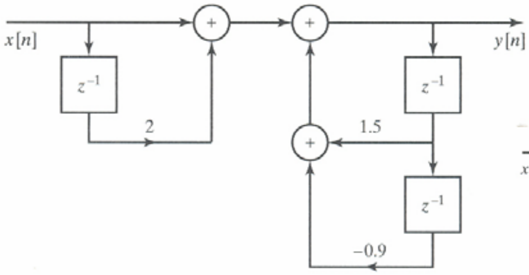


Figure 6.6 Direct form I implementation of Eq. (6.16).

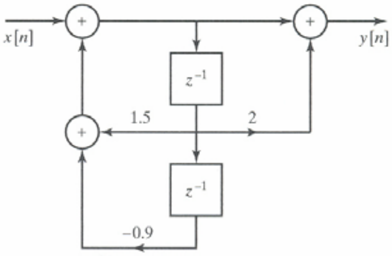


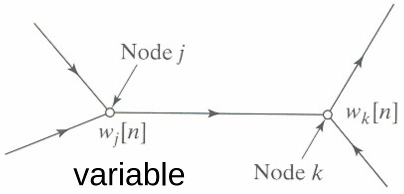
Figure 6.7 Direct form II implementation of Eq. (6.16).

Part II: Signal flow graph description

- Block diagram representation of computational structures
- Signal flow graph description
- Basic structures for IIR systems
- Transposed forms
- Basic structures for FIR systems

Signal flow graph (SFG)

- As an alternative to block diagrams with a few notational differences.
- A network of directed branches connecting nodes.



variable

Figure 6.8 Example of nodes and branches in a signal flow graph.

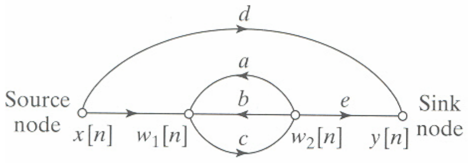
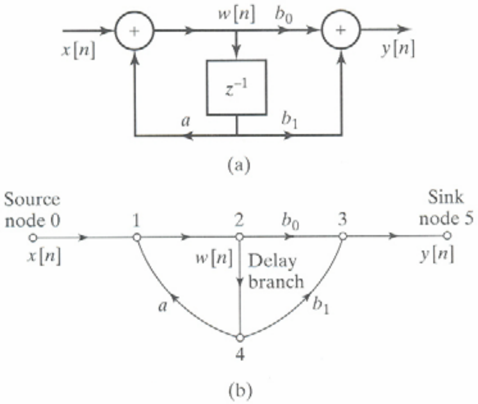
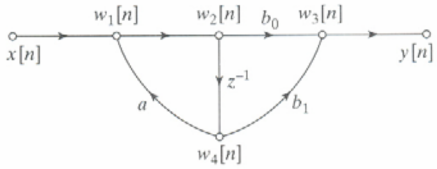


Figure 6.9 Example of a signal flow graph showing source and sink nodes.



(a)

(b)



Nodes in SFG represent both branching points and adders (depending on the number of incoming branches), while in the diagram a special symbol is used for adders and a node has only one incoming branch.

SGF is simpler to draw.

From flow graph to system function

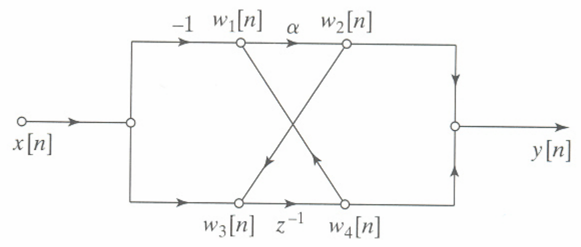


Figure 6.12 Flow graph not in standard direct form.

$$w_1[n] = w_4[n] - x[n]$$
$$w_2[n] = \alpha w_1[n]$$
$$w_3[n] = w_2[n] + x[n]$$
$$w_4[n] = w_3[n-1]$$
$$y[n] = w_2[n] + w_4[n]$$

- Not a direct form,
  - cannot obtain H(z) by inspection.
  - But can write an equation for each node
    - $w_4[n] = w_3[n-1]$ involve feedback, difficult to solve
    - By z-transform → linear equations

From flow graph to system function

$$\begin{cases} W_1(z) = W_4(z) - X(z) \\ W_2(z) = \alpha W_1(z) \\ W_3(z) = W_2(z) + X(z) \\ W_4(z) = z^{-1} W_3(z) \\ Y(z) = W_2(z) + W_4(z) \end{cases}$$

$$\begin{cases} W_2(z) = \alpha(W_4(z) - X(z)) \\ \\ W_4(z) = z^{-1}(W_2(z) + X(z)) \\ Y(z) = W_2(z) + W_4(z) \end{cases}$$

$$Y(z) = \left(\frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \right) X(z)$$

$$H(z) = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

If α is real, the system is ? All-pass

$$h[n] = \alpha^{n-1} u[n-1] - \alpha^{n+1} u[n]$$

Causal!

From flow graph to system function

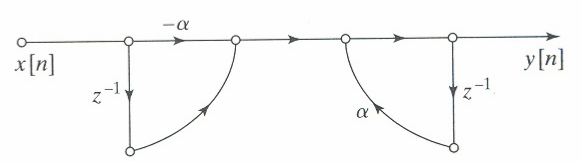


Figure 6.13 Direct form I equivalent of Figure 6.12.

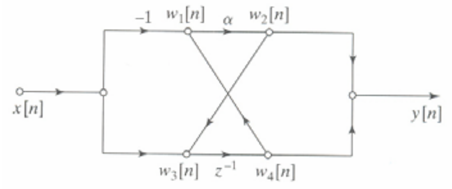


Figure 6.12 Flow graph not in standard direct form.

Different implementations, different amounts of computational resources

Part III: Basic structures for IIR systems

- Block diagram representation of computational structures
- Signal flow graph description
- Basic structures for IIR systems
- Transposed forms
- Basic structures for FIR systems

Direct form I

$$y[n]=\sum_{k=1}^{N}a_k y[n-k]+\sum_{k=0}^{M}b_k x[n-k]$$

$$H(z)=\frac{\sum_{k=0}^{M}b_k z^{-k}}{1-\sum_{k=1}^{N}a_k z^{-k}}$$

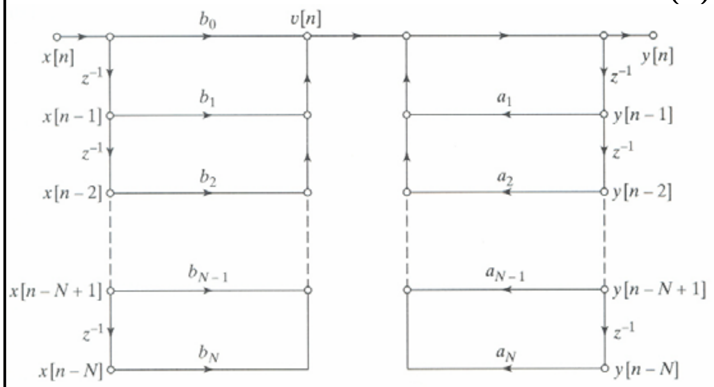


Figure 6.14 Signal flow graph of direct form I structure for an *N*th-order system.

Direct form II

$$y[n]=\sum_{k=1}^{N}a_k y[n-k]+\sum_{k=0}^{M}b_k x[n-k]$$

$$H(z)=\frac{\sum_{k=0}^{M}b_k z^{-k}}{1-\sum_{k=1}^{N}a_k z^{-k}}$$

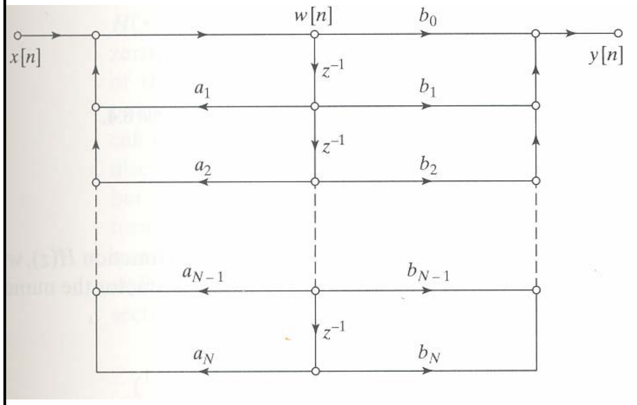


Figure 6.15 Signal flow graph of direct form II structure for an *N*th-order system.

Example

$$H(z)=\frac{1+2z^{-1}+z^{-2}}{1-0.752z^{-1}+0.125z^{-2}}$$

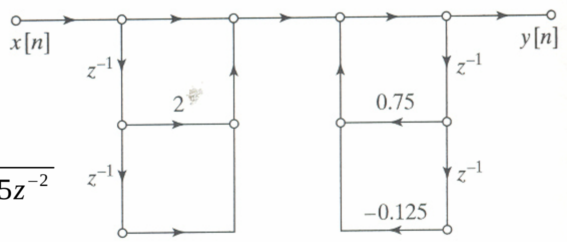


Figure 6.16 Direct form I structure for Example 6.4.

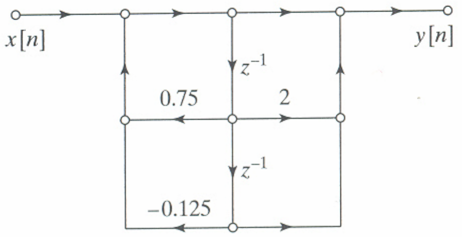


Figure 6.17 Direct form II structure for Example 6.4.

Cascade form

Factor the numerator and denominator polynomials

$$H(z)=\frac{\sum_{k=0}^{M}b_k z^{-k}}{1-\sum_{k=1}^{N}a_k z^{-k}}=A\frac{\prod_{k=1}^{M_1}(1-f_k z^{-1})\prod_{k=1}^{M_2}(1-g_k z^{-1})(1-g_k^* z^{-1})}{\prod_{k=1}^{N_1}(1-c_k z^{-1})\prod_{k=1}^{N_2}(1-d_k z^{-1})(1-d_k^* z^{-1})}$$

$$H(z)=\prod_{k=1}^{N_s}\frac{b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2}}{1-a_{1k}z^{-1}-a_{2k}z^{-2}}$$

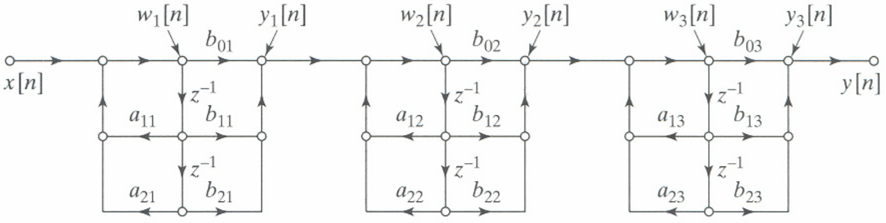
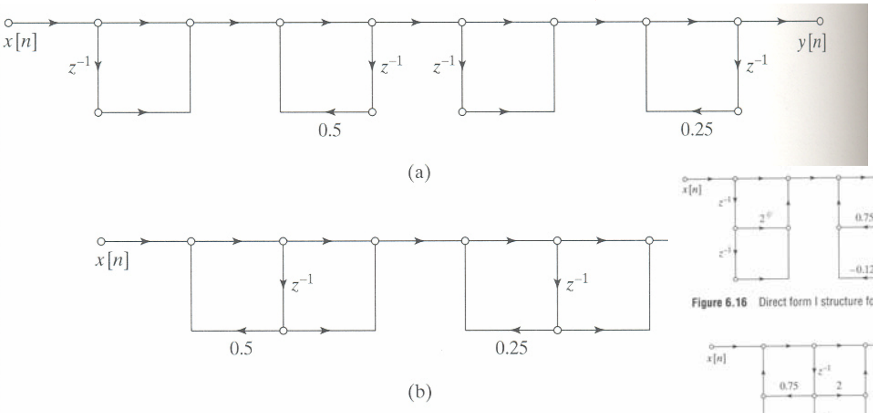


Figure 6.18 Cascade structure for a sixth-order system with a direct form II realization of each second-order subsystem.

An example: from 2nd-order to 1st-order cascade

$$H(z)=\frac{1+2z^{-1}+z^{-2}}{1-0.752z^{-1}+0.125z^{-2}}=\frac{(1+z^{-1})\cdot(1+z^{-1})}{(1-0.5z^{-1})\cdot(1-0.25z^{-1})}$$



(a)

(b)

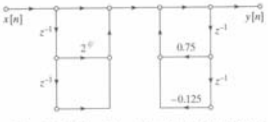


Figure 6.16 Direct form I structure for Example 6.4.

Figure 6.19 Cascade structures for Example 6.5. (a) Direct form I (b) Direct form II subsections.

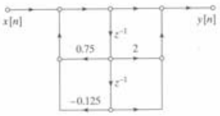


Figure 6.17 Direct form II structure for Example 6.4.

Parallel form by partial fraction expansion

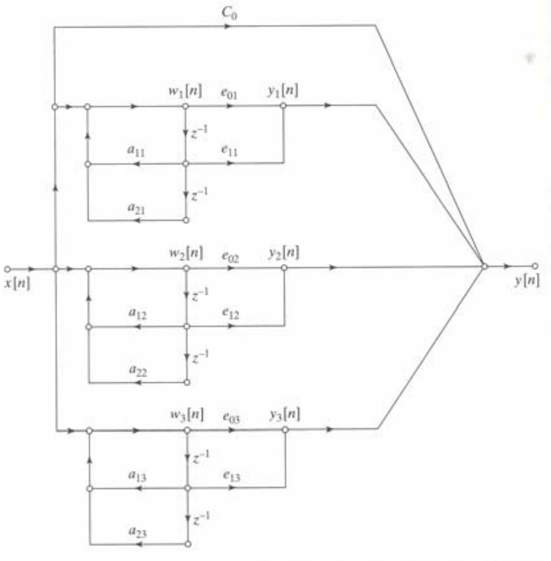


Figure 6.20 Parallel-form structure for sixth-order system ($M = N = 6$) with the real and complex poles grouped in pairs.

$$H(z)=\sum_{k=0}^{N_p} C_k z^{-k}+\sum_{k=1}^{N_1}\frac{A_k}{1-c_k z^{-1}}$$
$$+\sum_{k=1}^{N_2}\frac{B_k(1-e_k z^{-1})}{(1-d_k z^{-1})(1-d_k^* z^{-1})}$$

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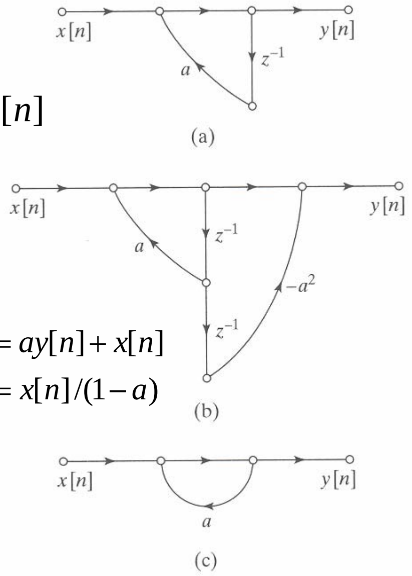
Feedback in IIR systems



$a^n u[n]$

$y[n] = ay[n] + x[n]$

$y[n] = x[n]/(1-a)$

Feedback loop: a closed path
Necessary but not sufficient condition for IIR system
(Feedback introduced poles could be cancelled by zeros)

$$H(z) = \frac{1-a^2 z^{-2}}{1-az^{-1}} = 1 + az^{-1}$$

All loops must contain at least one unit delay element

Figure 6.23 (a) System with feedback loop. (b) FIR system with feedback loop. (c) Noncomputable system.

Part IV: Transposed forms

- Block diagram representation of computational structures
- Signal flow graph description
- Basic structures for IIR systems
- Transposed forms
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Transposed form for a first-order system

Flow graph reversal or transposition also provides alternatives: reversing the directions of all branches and reversing the input and output

Resulting in same H(z)

$$H(z) = \frac{1}{1 - az^{-1}}$$

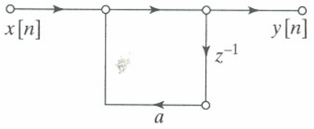


Figure 6.24 Flow graph of simple first-order system.

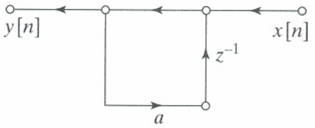


Figure 6.25 Transposed form of Figure 6.24.

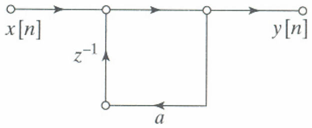


Figure 6.26 Structure of Figure 6.25 redrawn with input on left.

Transposed direct form II and direct form II

The transposed direct form II implements the zeros first and then the poles, being important effect for finite-precision existing

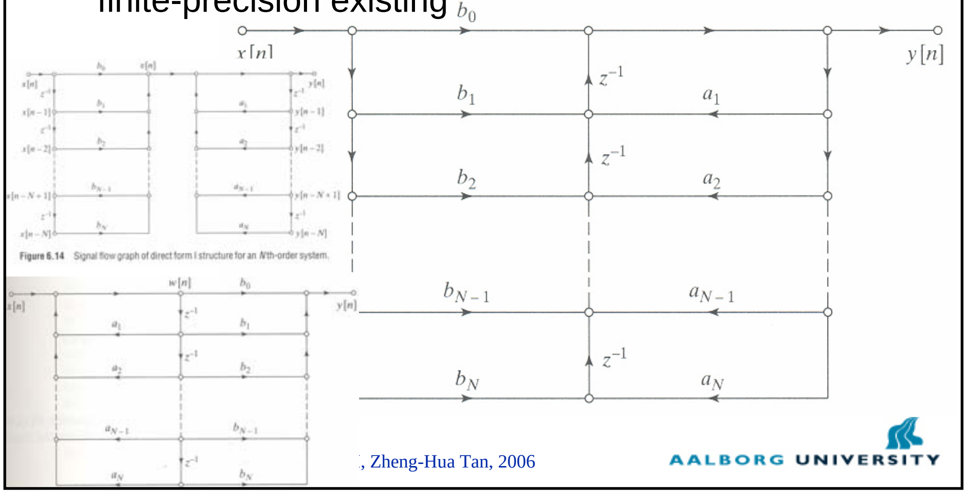


Figure 6.14 Signal flow graph of direct form I structure for an *N*th-order system.

Part V: Basic structures for FIR systems

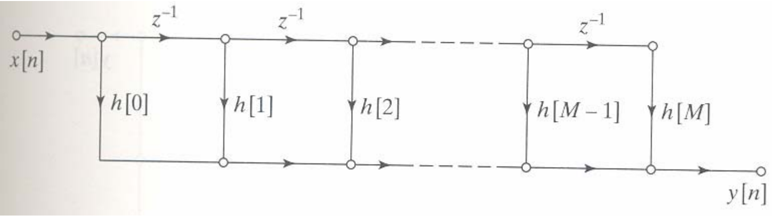
- Block diagram representation of computational structures
- Signal flow graph description
- Basic structures for IIR systems
- Transposed forms
- Basic structures for FIR systems

Direct form

- So far, system function has both poles and zeros. FIR systems as a special case.
- Causal FIR system function has only zeros (except for poles as z=0)

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \qquad h[n] = \begin{cases} b_n, & n = 0,1,\ldots,M \\ 0, & \text{otherwise} \end{cases}$$

- Form I and form II are the same.



Cascade form

Factoring the polynomial system function

$$H(z)=\sum_{n=0}^{M}h[n]z^{-n}=\prod_{k=1}^{M_s}(b_{0k}+b_{1k}z^{-1}+b_{2k}z^{-2})$$

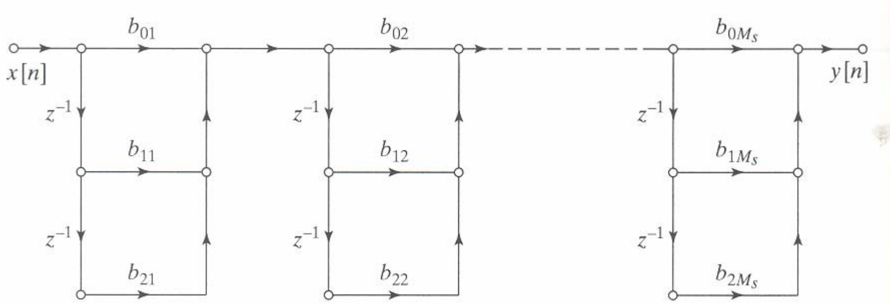


Figure 6.33 Cascade-form realization of an FIR system.

Linear-phase FIR systems

- Generalized linear-phase system

$$H(e^{j\omega})=A(e^{j\omega})e^{-j\omega\alpha+j\beta}$$

$A(e^{j\omega})$ is a real function of ω,

α and β are real constants

- *Causal FIR systems* have generalized linear-phase if h[n] satisfies the symmetry condition

$$h[M-n]=h[n],\ \ n=0,1,\ldots,M$$

or

$$h[M-n]=-h[n],\ \ n=0,1,\ldots,M$$

$$y[n]=\sum_{k=0}^{M}b_k x[n-k]$$

Linear-phase FIR systems

if M is an even integer

$$y[n]=\sum_{k=0}^{M}h[k]x[n-k]$$

$$=\sum_{k=0}^{M/2-1}h[k]x[n-k]+h[k/M]x[n-M/2]+\sum_{k=M/2+1}^{M}h[k]x[n-k]$$

$$=\sum_{k=0}^{M/2-1}h[k]x[n-k]+h[k/M]x[n-M/2]+\sum_{k=0}^{M/2-1}h[M-k]x[n-M+k]$$

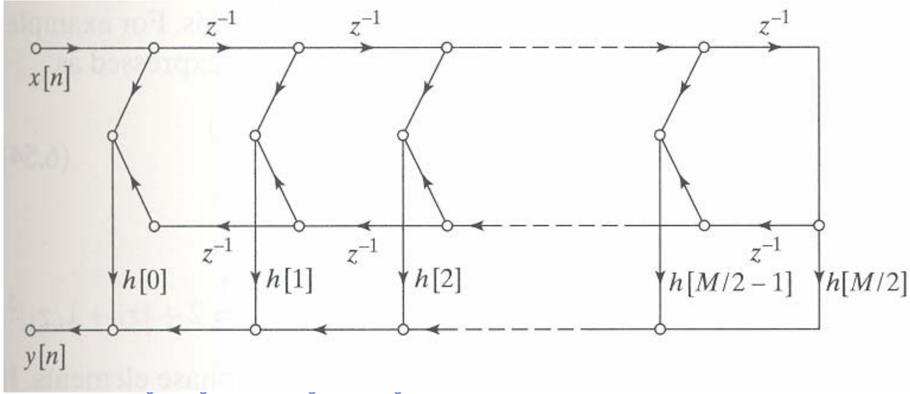
if $h[M-n]=h[n]$

$$y[n]=\sum_{k=0}^{M/2-1}h[k]\underline{(x[n-k]+x[n-M+k])}+h[k/M]x[n-M/2]$$

Linear phase FIR systems

M is an even integer and h[M-n]=h[n]

$$y[n]=\sum_{k=0}^{M/2-1}h[k](x[n-k]+x[n-M+k])+h[k/M]x[n-M/2]$$



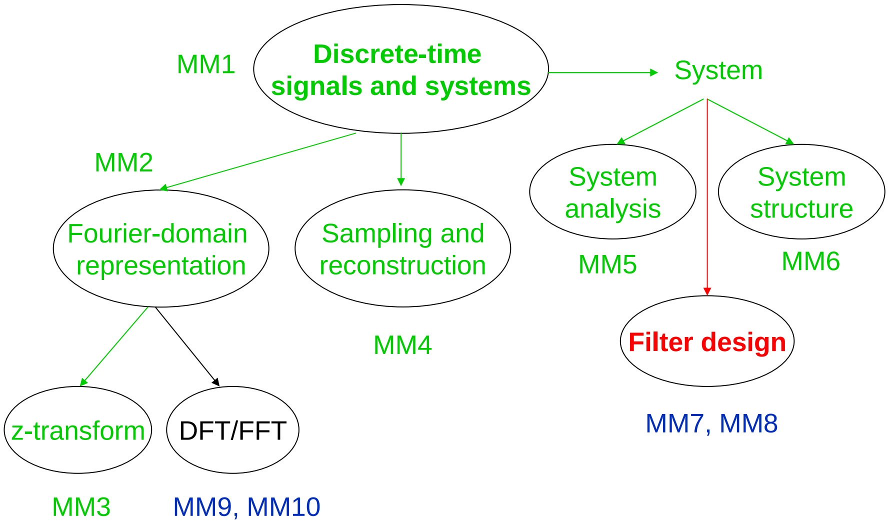
Discussions

- Implementation of FIR and IIR systems
- Use signal block diagram flow graph representation to show the computational structures
- Although two structures may have equivalent input-output charateristics for infinite-precision represenations of coefficients and variables, they may have dramatically different behaviour when the numerical precision is limited.

Summary

- Block diagram representation of computational structures
- Signal flow graph description
- Basic structures for IIR systems
- Transposed forms
- Basic structures for FIR systems

Course at a glance



MM1 Discrete-time signals and systems

System

MM2 Fourier-domain representation

Sampling and reconstruction

MM4

System analysis

MM5

System structure

MM6

Filter design

MM7, MM8

z-transform

MM3

DFT/FFT

MM9, MM10