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Lecture 5: System analysis

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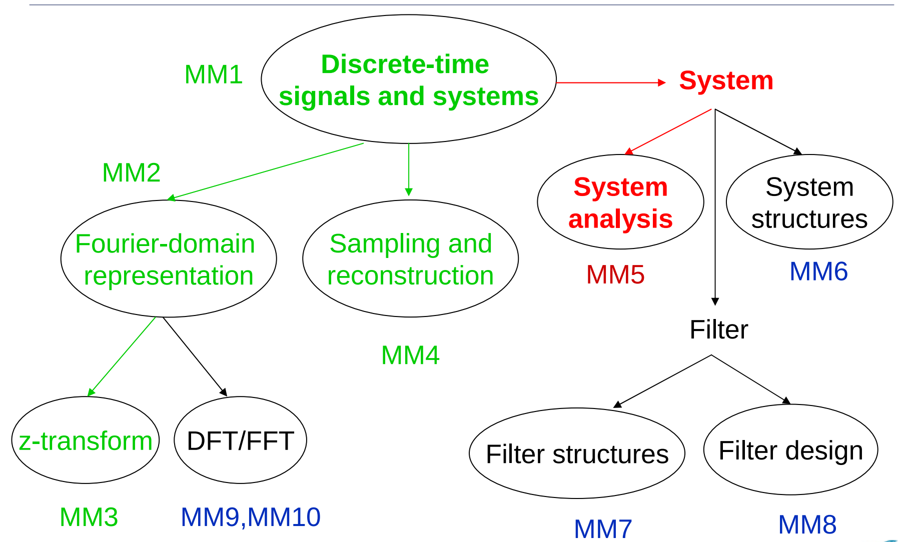
Course at a glance



System analysis

- Three domains
  - Time domain: impulse response, convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  - Frequency domain: frequency response

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

  - z-transform: system function

$$Y(z) = X(z)H(z)$$

- LTI system is completed characterized by …

Part I: Frequency response

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

Frequency response

- Relationship btw Fourier transforms of input and output

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- In polar form
  - Magnitude → magnitude response, gain, distortion

$$|Y(e^{j\omega})| = |X(e^{j\omega})| \cdot |H(e^{j\omega})|$$

  - Phase → phase response, phase shift, distortion

$$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$$

Ideal lowpass filter – an example

- Frequency response

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

  - Frequency selective filter
- Impulse response

$$h_{lp}[n] = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

  - Noncausal, cannot be implemented! $h[n] \overset{?}{=} 0, \quad n < 0$
  - How to make a noncausal system causal?

Make noncausal system causal

- Cascading systems
  - Ideal delay $h[n] = \delta[n - n_d]$

$$x[n] \longrightarrow \underset{h[n]=\delta[n+1]-\delta[n]}{\text{Forward difference}} \longrightarrow \underset{h[n]=\delta[n-1]}{\text{One-sameple delay}} \longrightarrow y[n]$$

$$\Downarrow$$

$$x[n] \longrightarrow \underset{h[n]=\delta[n]-\delta[n-1]}{\text{Backward difference}} \longrightarrow y[n]$$

  - In general, any noncausal FIR system can be made cause by cascading it with a sufficiently long delay!
  - But ideal lowpass filter is an IIR system!

Phase distortion and delay

- Ideal delay system

$$h_{id}[n] = \delta[n - n_d]$$

Delay distortion

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

$$|H_{id}(e^{j\omega})| = 1$$

$$\angle H_{id}(e^{j\omega}) = -\omega n_d, |\omega| < \pi$$

Linear phase distortion

- Ideal lowpass filter with linear phase

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

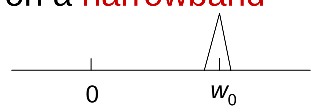
$$h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \qquad -\infty < n < \infty$$

Ideal lowpass filter is always noncausal!

Group delay

- A measure of the linearity of the phase
- Concerning the phase distortion on a narrowband signal

$$x[n] = s[n]\cos(\omega_0 n)$$



- For this input with spectrum only around w_0, phase effect can be approximated around w_0 as the linear approximation (though in reality maybe nonlinear)

$$\angle H(e^{j\omega}) \approx -\omega n_d - \phi_0$$

and the output is approximately

$$y[n] \approx |H(e^{j\omega_0})|\, s[n-n_d]\cos(\omega_0(n-n_d)-\phi_0)$$

- Group delay $\tau = grd[H(e^{j\omega})] = -\dfrac{d}{d\omega}\{\arg[H(e^{j\omega})]\}$

An example of group delay

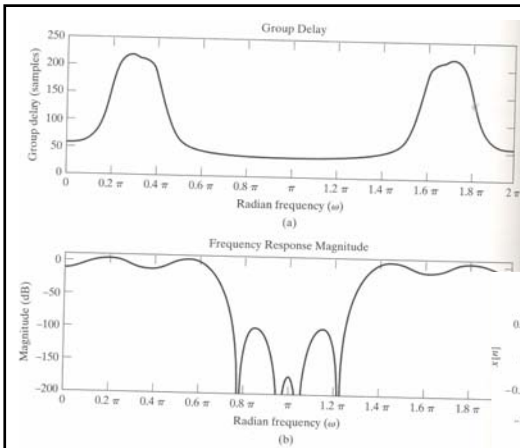


Figure 5.1 Frequency response magnitude and group delay for the filter in Example 5.1.

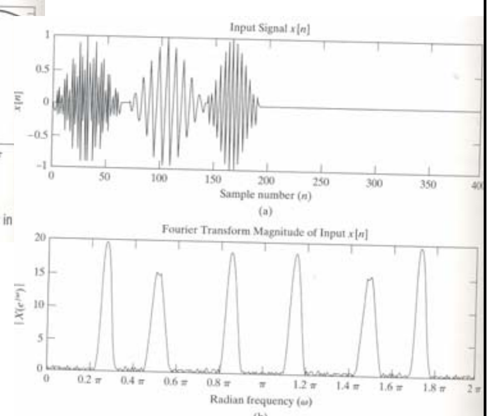


Figure 5.2 Input signal and associated Fourier transform magnitude for Example 5.1.

An example of group delay

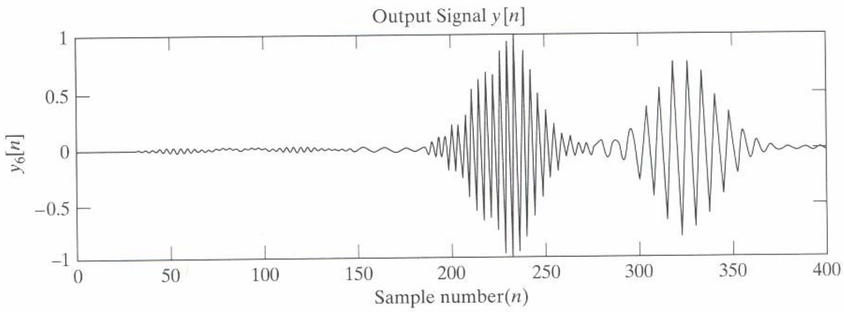


Figure 5.3 Output signal for Example 5.1.

Since the filter has considerable attenuation at $\omega = 0.85\pi$, the pulse at that frequency is not clearly present in the output. Also, since the group delay at $\omega = 0.25\pi$ is approximately 200 samples and at $\omega = 0.5\pi$ is approximately 50 samples, the second pulse in $x[n]$ will be delayed by about 200 samples and the third pulse by 50 samples, as we see is the case in Figure 5.3.

Part II: System functions

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

System function of LCCDE systems

- Linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

- z-transform format

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \left(\frac{b_0}{a_0}\right) \frac{\prod_{m=1}^{M}(1-c_m z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

$(1-c_m z^{-1})$ in the numerator

a zero at $z = c_m$ a pole at $z = 0$

$(1-d_k z^{-1})$ in the denominator

a zero at $z = 0$ a pole at $z = d_k$

Stability and causality

- Stable
  - h[n] absolutely summable
  - H(z) has a ROC including the unit circle
- Causal
  - h[n] right side sequence
  - H(z) has a ROC being outside the outermost pole

Inverse systems

- Many systems have inverses, specially systems with rational system functions

$$G(z) = H(z)H_i(z) = 1$$

$$H_i(z) = \frac{1}{H(z)}$$

$$g[n] = h[n] * h_i[n] = \delta[n]$$

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{m=1}^{M}(1-c_m z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})}$$

$$H_i(z) = \left(\frac{a_0}{b_0}\right)\frac{\prod_{k=1}^{N}(1-d_k z^{-1})}{\prod_{m=1}^{M}(1-c_m z^{-1})}$$

  - Poles become zeros and vice versa.
  - ROC: must have overlap btw the two for the sake of G(z).

Example

$$H(z) = \frac{1-0.5z^{-1}}{1-0.9z^{-1}}, |z| > 0.9$$

$$H_i(z) = \frac{1-0.9z^{-1}}{1-0.5z^{-1}} = \frac{1}{1-0.5z^{-1}} - \frac{0.9z^{-1}}{1-0.5z^{-1}}$$

So, $|z| > 0.5$

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1]$$

Part III: Magnitude and phase

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

Relationship btw magnitude and phase

- In particular, for systems with rational system functions, there is constraint btw magnitude and phase

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

- Consider the square of the magnitude

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(z)H^*(1/z^*)|_{z=e^{j\omega}}$$

$$H(z) = (\frac{b_0}{a_0})\frac{\prod_{m=1}^{M}(1-c_m z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad H^*(1/z^*) = (\frac{b_0}{a_0})\frac{\prod_{m=1}^{M}(1-c_m^* z)}{\prod_{k=1}^{N}(1-d_k^* z)}$$

$$C(z) = H(z)H^*(1/z^*) = (\frac{b_0}{a_0})^2\frac{\prod_{m=1}^{M}(1-c_m z^{-1})(1-c_m^* z)}{\prod_{k=1}^{N}(1-d_k z^{-1})(1-d_k^* z)}$$

Example 5.11 Systems with the Same $C(z)$

Consider two stable systems with system functions

$$H_1(z) = \frac{2(1 - z^{-1})(1 + 0.5z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})} \tag{5.83}$$

and

$$H_2(z) = \frac{(1 - z^{-1})(1 + 2z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}. \tag{5.84}$$

The pole–zero plots for these systems are shown in Figures 5.19(a) and 5.19(b), respectively.

Now,

$$\begin{aligned} C_1(z) &= H_1(z)H_1^*(1/z^*) \\ &= \frac{2(1 - z^{-1})(1 + 0.5z^{-1})2(1 - z)(1 + 0.5z)}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z)(1 - 0.8e^{j\pi/4}z)} \end{aligned} \tag{5.85}$$

and

$$\begin{aligned} C_2(z) &= H_2(z)H_2^*(1/z^*) \\ &= \frac{(1 - z^{-1})(1 + 2z^{-1})(1 - z)(1 + 2z)}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z)(1 - 0.8e^{j\pi/4}z)}. \end{aligned} \tag{5.86}$$

An example

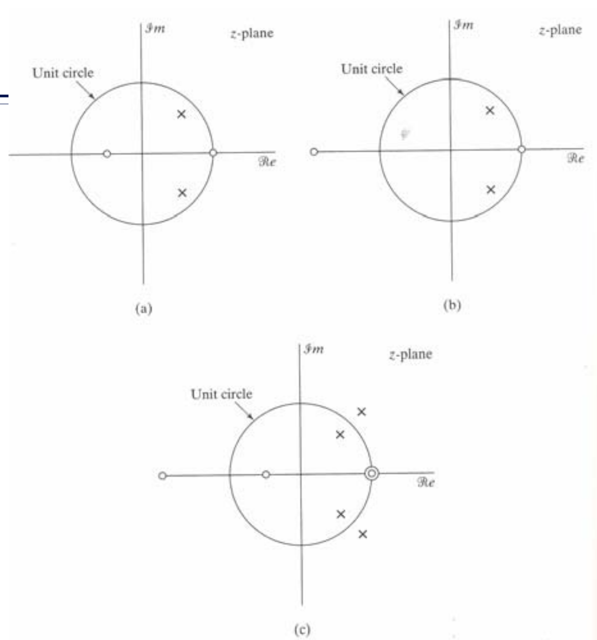


Figure 5.19 Pole–zero plots for two system functions and their common magnitude-squared function. (a) $H_1(z)$. (b) $H_2(z)$. (c) $C_1(z)$, $C_2(z)$.

Part VI: All-pass systems

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

All-pass systems

- Consider the following stable system function

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}$$

$$= e^{-j\omega} \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

- $|H_{ap}(e^{j\omega})| = 1$ all-pass system: for which the frequency response magnitude is a constant.
- General form

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

An example

P275 Example 5.13, First-order all-pass system

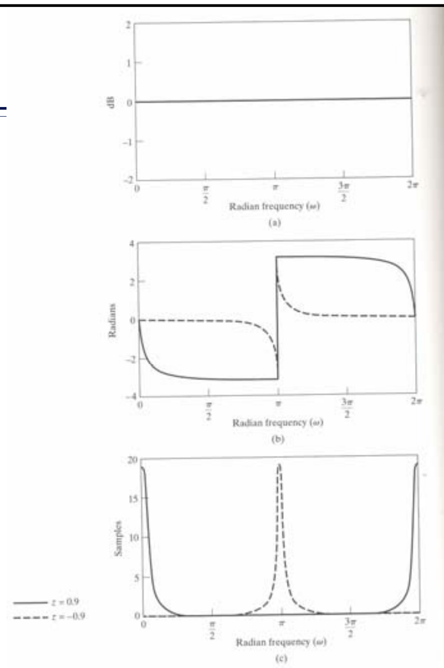


Figure 5.22 Frequency response for all-pass filters with real poles at $z = 0.9$ (solid line) and $z = -0.9$ (dashed line). (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

An example

Second-order all-pass system

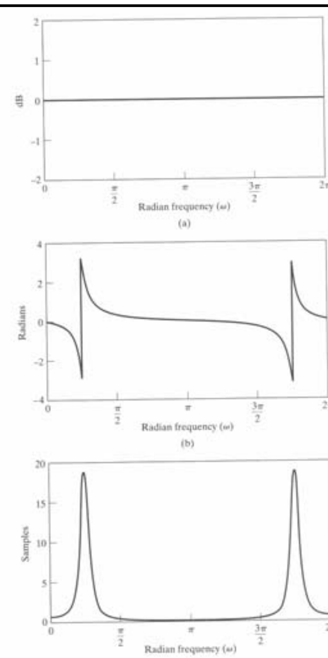


Figure 5.23 Frequency response of second-order all-pass system with poles at $z = 0.9e^{\pm j\pi/4}$. (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

Part V: Minimum-phase systems

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

Minimum-phase systems

- Magnitude does not uniquely characterize the system
  - Stable and causal → poles inside unit circle, no restriction on zeros
  - Zeros are also inside unit circle → inverse system is also stable and causal (in many situations, we need inverse systems!)
  - → such systems are called minimum-phase systems (explanation to follow): are stable and causal and have stable and causal inverses

Minimum-phase and all-pass decomposition

Any rational system function can be expressed as:

$$H(z) = H_{\min}(z) H_{ap}(z)$$

Suppose H(z) has one zero outside the unit circle at $z = 1/c^*, |c| < 1$

$$H(z) = H_1(z)(z^{-1} - c^*)$$

$$= H_1(z)(1 - cz^{-1}) \frac{z^{-1} - c^*}{1 - cz^{-1}}$$

minimum-phase all-pass

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Frequency response compensation

When the distortion system is not minimum-phase system:

$$H_d(z) = H_{d\min}(z) H_{ap}(z) \quad H_c(z) = \frac{1}{H_{d\min}(z)}$$

$$G(z) = H_d(z) H_c(z) = H_{ap}(z)$$

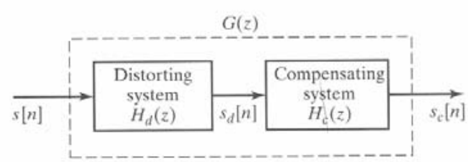

Figure 5.25 Illustration of distortion compensation by linear filtering.

Frequency response magnitude is compensated

Phase response is the phase of the all-pass

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Properties of minimum-phase systems

- From minimum-phase and all-pass decomposition

$$H(z) = H_{\min}(z)H_{ap}(z)$$

$$\arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

- From previous figures, the continuous-phase curve of an all-pass system is negative for $0 \le \omega \le \pi$
- So change from minimum-phase to nonminimum-phase (+all-pass phase) always decreases the continuous phase or increases the negative of the phase (called the phase-lag function). Minimum-phase is more precisely called minimum phase-lag system

Part VI: Linear-phase systems

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

Design a system with non-zero phase

- System design sometimes desires
  - Constant frequency response magnitude
  - Zero phase, when not possible
    - accept phase distortion, in particular linear phase since it only introduce time shift
    - Nonlinear phase will change the shape of the input signal though having constant magnitude response

Ideal delay

$$H_{id}(e^{j\omega}) = e^{-j\omega\alpha}, |\omega| < \pi$$

$$|H_{id}(e^{j\omega})| = 1$$

$$\angle H_{id}(e^{j\omega}) = -\omega\alpha, |\omega| < \pi$$

$$grd[H_{id}(e^{j\omega})] = \alpha$$

$$h_{id}[n] = \frac{\sin \pi(n-\alpha)}{\pi(n-\alpha)}$$

when $\alpha = n_d$

$$h_{id}[n] = \delta[n - n_d]$$

Ideal lowpass with linear phase

$$h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi(n - n_d)}$$

Generalized linear phase

- Linear phase filters

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}$$

- Generalized linear phase filters

$$H(e^{j\omega}) = A(e^{j\omega}) e^{-j\omega\alpha + j\beta}$$

$A(e^{j\omega})$ is a real function of ω,

α and β are real constants

Summary

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

Course at a glance