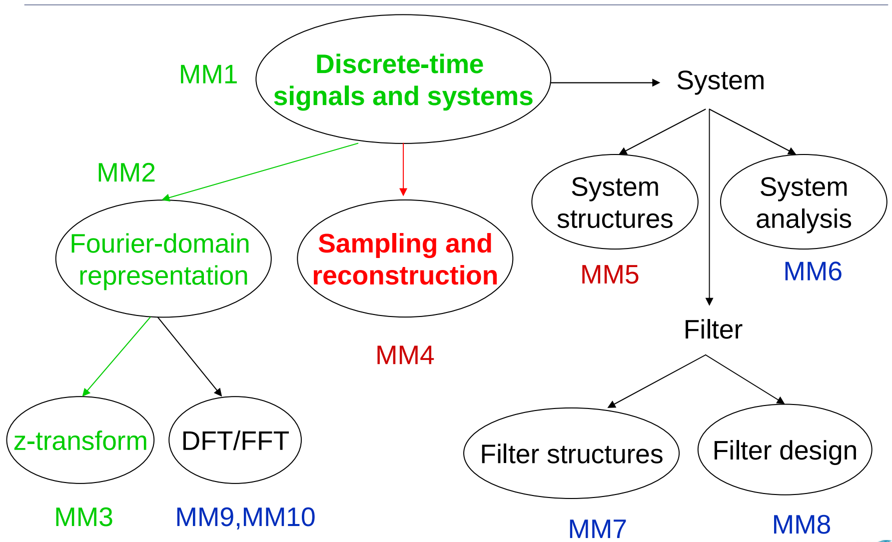
Digital Signal Processing, Fall 2006

Lecture 4: Sampling and reconstruction

Zheng-Hua Tan

Department of Electronic Systems
Aalborg University, Denmark
zt@kom.aau.dk

Course at a glance



Part I: Periodic sampling

- Periodic sampling
- Frequency domain representation
- Reconstruction
- Changing the sampling rate using discrete-time processing

Periodic sampling

- From continuous-time $x_c(t)$ to discrete-time $x[n]$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

  - Sampling period T
  - Sampling frequency $f_s = 1/T$

    $\Omega_s = 2\pi / T$

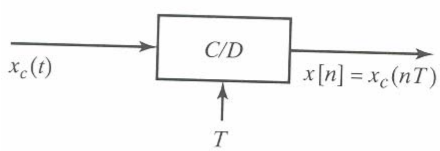


Figure 4.1 Block diagram representation of an ideal continuous-to-discrete-time (C/D) converter.

Two stages

In practice?

- Mathematically
  - Impulse train modulator
  - Conversion of the impulse train to a sequence

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_s(t) = x_c(t)s(t)$$

$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$

$$x_c(t) = \int_{-\infty}^{\infty} x_c(\tau)\delta(t-\tau)d\tau$$

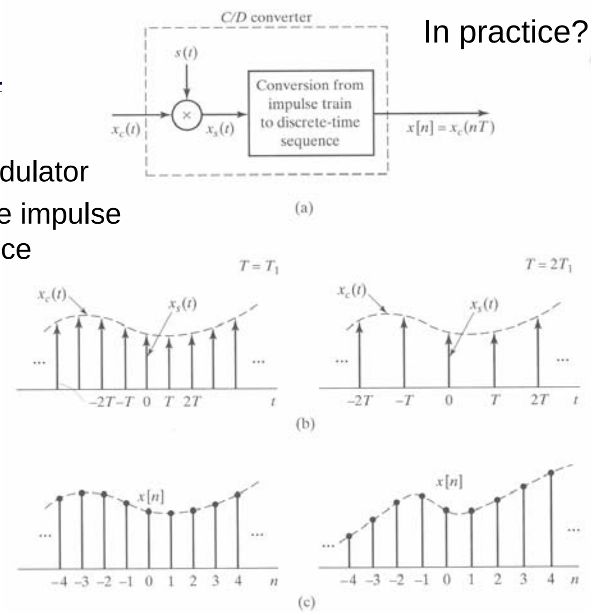


Figure 4.2 Sampling with a periodic impulse train followed by conversion to a discrete-time sequence. (a) Overall system. (b) $x_s(t)$ for two sampling rates. (c) The output sequence for the two different sampling rates.

Periodic sampling

- Tow-stage representation
  - Strictly a mathematical representation that is convenient for gaining insight into sampling in both the time and frequency domains.
  - Physical implementation is different.
  - $x_s(t)$ a continuous-time signal, an impulse train, zero except at nT
  - $x[n]$ a discrete-time sequence, time normalization, no explicit information about sampling rate
- Many-to-many → in general not invertible

Part II: Frequency domain represent.

- Periodic sampling
- Frequency domain representation
- Reconstruction
- Changing the sampling rate using discrete-time processing

Frequency-domain representation

- From $x_c(t)$ to $x_s(t)$

The Fourier transform of a periodic impulse train is a periodic impulse train.

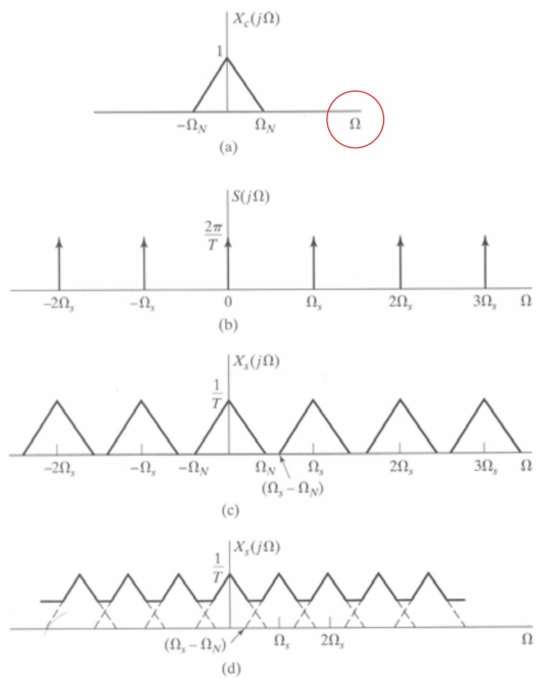
$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \quad \leftrightarrow \quad S(j\Omega) \overset{?}{=} \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s)$$

$$x_s(t) = x_c(t)s(t) \quad \leftrightarrow \quad X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * S(j\Omega)$$

$$= x_c(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \qquad = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\Omega - k\Omega_s))$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$$

- The Fourier transform of $x_s(t)$ consists of periodic repetition of the Fourier transform of $x_c(t)$.



$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$\Omega_s - \Omega_N > \Omega_N \text{ or } \Omega_s > 2\Omega_N$$

Figure 4.3 Effect in the frequency domain of sampling in the time domain. (a) Spectrum of the original signal. (b) Spectrum of the sampling function. (c) Spectrum of the sampled signal with $\Omega_s > 2\Omega_N$. (d) Spectrum of the sampled signal with $\Omega_s < 2\Omega_N$.

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Recovery

$$X_r(j\Omega) = H_r(j\Omega) X_s(j\Omega)$$

Ideal lowpass filter with gain T and cutoff frequency Ω_c

$$\Omega_N < \Omega_c < (\Omega_s - \Omega_N)$$

$$X_r(j\Omega) = X_c(j\Omega)$$

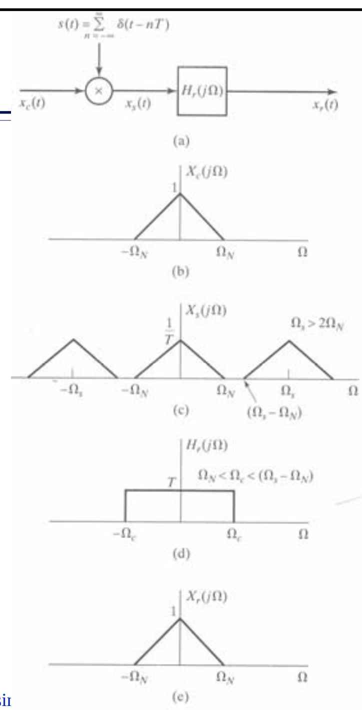
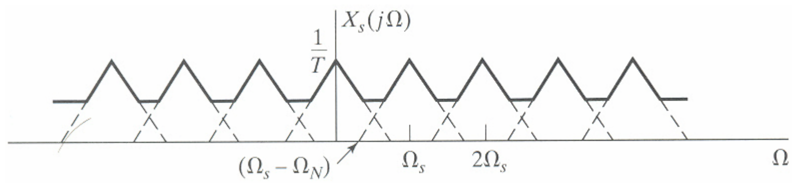


Figure 4.4 Exact recovery of a continuous-time signal from its samples using an ideal lowpass filter.

Aliasing distortion

- Due to the overlap among the copies of $X_c(j\Omega)$, due to $\Omega_s \le 2\Omega_N$
- $X_c(j\Omega)$ not recoverable by lowpass filtering



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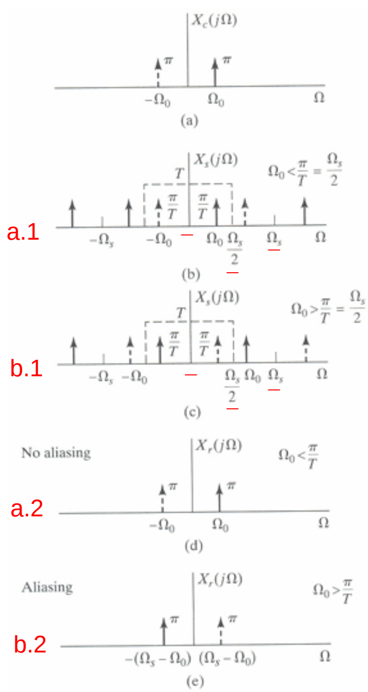

Aliasing – an example

$$x_c(t) = \cos\Omega_0 t$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$x_r(t) = \cos\Omega_0 t$$

$$x_r(t) = \cos(\Omega_s - \Omega_0)t$$

Figure 4.5 The effect of aliasing in the sampling of a cosine signal.

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Nyquist sampling theorem

Given bandlimited signal $x_c(t)$ with

$$X_c(j\Omega) = 0, \quad \text{for } |\Omega| \geq \Omega_N$$

Then $x_c(t)$ is uniquely determined by its samples

$$x[n] = x_c(nT), \qquad -\infty < n < \infty$$

If

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$$

Ω_N is called Nyquist frequency

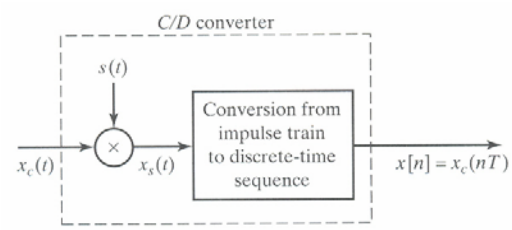
$2\Omega_N$ is called Nyquist rate

Fourier transform of x[n]

From $x_s(t)$ to $x[n]$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

$$x[n] = x_c(nT), \quad -\infty < n < \infty$$



From $X_s(j\Omega)$ to $X(e^{j\omega})$

By taking continuous-time Fourier transform of $x_s(t)$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\Omega Tn} \qquad \left(X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \right)$$

By taking discrete-time Fourier transform of x[n]

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad X_s(j\Omega) = X(e^{j\omega})|_{\omega = \Omega T} = X(e^{j\Omega T})$$

Fourier transform of x[n]

$$X_s(j\Omega) = X(e^{j\omega})|_{\omega=\Omega T} = X(e^{j\Omega T})$$

$$X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega - 2\pi k}{T}\right)$$

- $X(e^{j\omega})$ is simply a frequency-scaled version of $X_s(j\Omega)$ with $\omega = \Omega T$
- $x_s(t)$ retains a spacing between samples equal to the sampling period T while $x[n]$ always has unity space.

Sampling and reconstruction of Sin Signal

$$x_c(t) = \cos(4000\pi t) \rightarrow \Omega_0 = 4000\pi$$

$$T = 1/6000 \rightarrow \Omega_s = 2\pi/T = 12000\pi \quad \therefore \text{ no aliasing}$$

$$x[n] = x_c(nT) = \cos(4000\pi nT) = \cos((2\pi/3)n) = \cos(\omega_0 n)$$

$$x_c(t) \leftrightarrow X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi) \qquad X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T} = X_s(j\omega/T) \text{ with normalized frequency } \omega = \Omega T$$

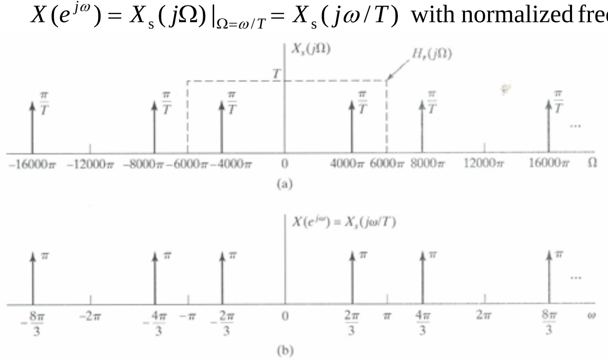


Figure 4.6 Continuous-time (a) and discrete-time (b) Fourier transforms for sampled cosine signal with frequency $\Omega_0 = 4000\pi$ and sampling period $T = 1/6000$.

How about $x_c(t) = \cos(16000\pi t)$

Part III: Reconstruction

- Periodic sampling
- Frequency domain representation
- Reconstruction
- Changing the sampling rate using discrete-time processing

Requirement for reconstruction

- On the basis of the sampling theorem, samples represent the signal exactly when:
  - Bandlimited signal
  - Enough sampling frequency
  - + knowledge of the sampling period → recover the signal

Reconstruction steps

(1) Given x[n] and T, the impulse train is

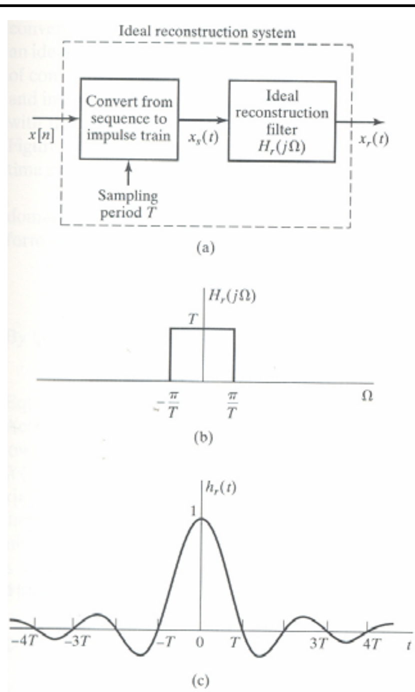
$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$$

i.e. the *n*th sample is associated with the impulse at t=nT.

(2) The impulse train is filtered by an ideal lowpass CT filter with impulse response $h_r(t) \leftrightarrow H_r(j\Omega)$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n) h_r(t-nT)$$

$$X_r(j\Omega) = H_r(j\Omega) X(e^{j\Omega T})$$

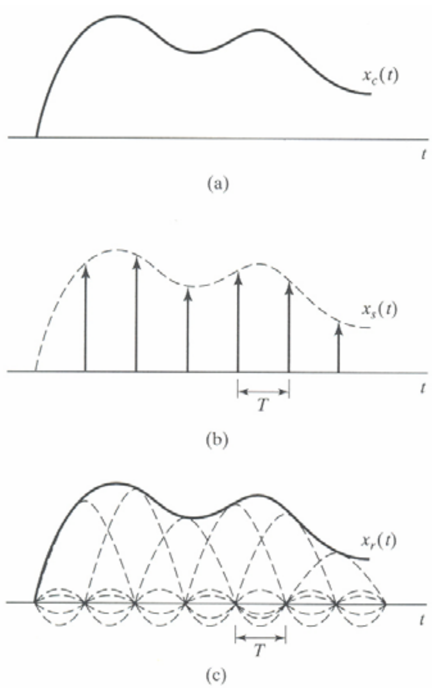


Commonly choose cutoff frequncy as

$$\Omega_c = \Omega_s / 2 = \pi / T$$

$$h_r(t) = \frac{\sin(\pi t / T)}{\pi t / T}$$

Figure 4.8 (a) Block diagram of an ideal bandlimited signal reconstruction system. (b) Frequency response of an ideal reconstruction filter. (c) Impulse response of an ideal reconstruction filter.



CT signal

Modulated impulse train

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$

Figure 4.9 Ideal bandlimited interpolation.

Ideal discrete-to-continuous-time converter

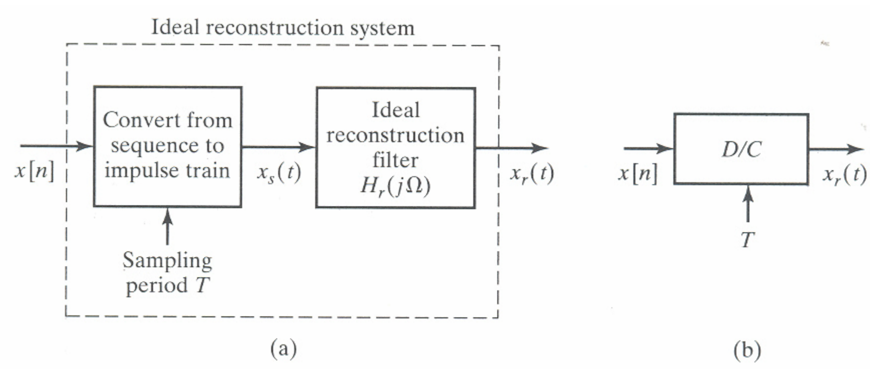
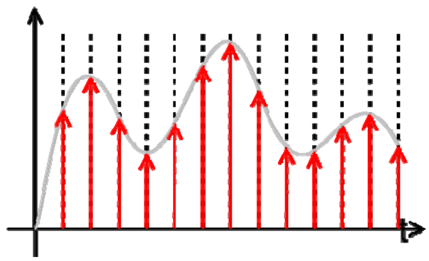


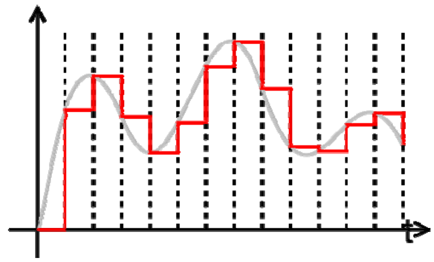
Figure 4.10 (a) Ideal bandlimited signal reconstruction. (b) Equivalent representation as an ideal D/C converter.

discrete-to-continuous-time converter

"Practical DACs do not output a sequence of dirac impulses (that, if ideally low-pass filtered, result in the original signal before sampling) but instead output a sequence of piecewise constant values or rectangular pulses"



Ideally sampled signal.



Piecewise constant signal typical of a practical DAC output.

From http://en.wikipedia.org/wiki/Digital-to-analog_converter.

Applications

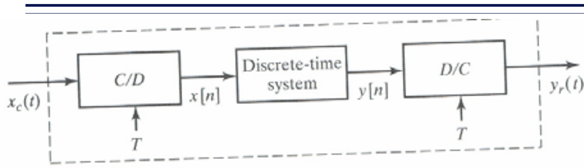


Figure 4.11 Discrete-time processing of continuous-time signals.

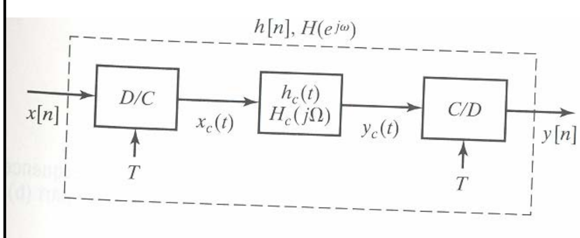


Figure 4.16 Continuous-time processing of discrete-time signals.

Part IV: Changing the sampling rate

- Periodic sampling
- Frequency domain representation
- Reconstruction
- Changing the sampling rate using discrete-time processing

Downsampling

$$x[n] = x_c(nT)$$

$$x'[n] = x_c(nT')$$

By reconstruction & re-sampling though not desirable

Using DT processing only:

- Sampling rate reduction by an integer factor – downsampling by "sampling" it

$$x_d[n] = x[nM] = x_c(nMT)$$

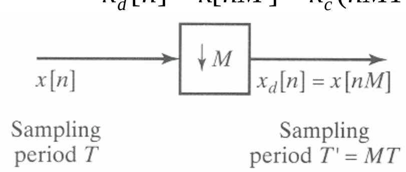


Figure 4.20 Representation of a compressor or discrete-time sampler.

Frequency domain

- DT Fourier transform $\quad x_d[n] = x[nM] = x_c(nMT)$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

$$X_d(e^{j\omega}) = \frac{1}{T'}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T'} - \frac{2\pi r}{T'}\right)\right)$$

$$= \frac{1}{MT}\sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right)$$

$$r = i + kM, \quad -\infty < k < \infty, \; 0 \le i \le M-1, \; -\infty < r < \infty$$

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT}\right)\right)\right]$$

Since $\quad X(e^{j(\omega - 2\pi i)/M}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi i}{MT} - \frac{2\pi k}{T}\right)\right)$

$$\rightarrow \quad X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega - 2\pi i)/M})$$

Similar to the Eq. above!

Frequency domain

- Sampling results in copies at $n\Omega_s = 2n\pi / T$
- Same, downsampling generates M copies of $X(e^{j\omega})$ with frequency scaled by M and shifted.
- Aliasing can be avoided if $X(e^{j\omega})$ is bandlimited

$$X(e^{j\omega}) = 0, \quad \omega_N \le |\omega| \le 2\pi$$

and $\quad 2\pi / M \ge 2\omega_N$

This example $\Omega_s = 4\Omega_N$

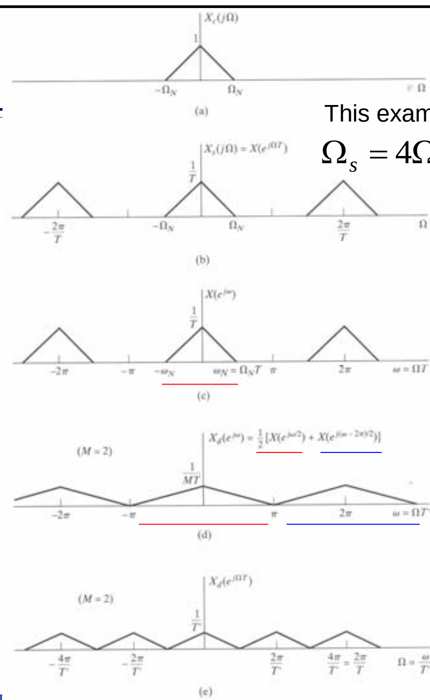


Figure 4.21 Frequency-domain illustration of downsampling.

Frequency domain

Downsampling factor is too large, causing aliasing, resulting in the need for DT ideal lowpass filter with cutoff frequency pi/M

To avoid aliasing in downsampling by a factor of M requires that

$$\omega_N M < \pi$$

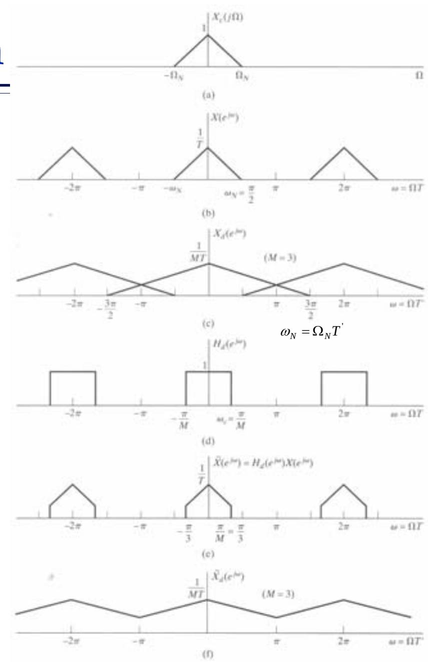


Figure 4.22 (a)–(c) Downsampling with aliasing. (d)–(f) Downsampling with prefiltering to avoid aliasing.

A general downsampling system

- The system is also called decimator
- The process is called decimation

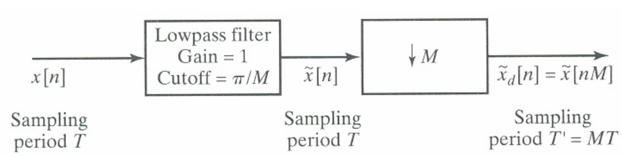


Figure 4.23 General system for sampling rate reduction by M.

Increasing sampling rate – upsampling

- Downsampling → analogous to sampling a CT signal
- Upsampling → analogous to D/C conversion

$$x[n] = x_c(nT)$$

$$x_i[n] = x_c(nT'), \qquad T' = T/L$$

$$x_i[n] = x[n/L] = x_c(nT/L), \qquad n = 0, \pm L, \pm 2L, \ldots$$

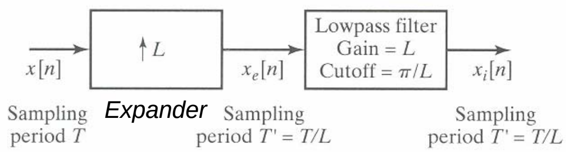


Figure 4.24 General system for sampling rate increase by L.

$$x_e[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

Fourier domain

- Fourier transform of the output of expander

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]\right)e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega Lk} = X(e^{j\omega L})$$

Which is s frequency scaled version, w is replaced by wL so

$$\omega = \Omega T'$$

An example

DTFT of $x[n] = x_c(nT)$

$$X_e(e^{j\omega}) = X(e^{j\omega L})$$

System: interpolator
Process: interpolation

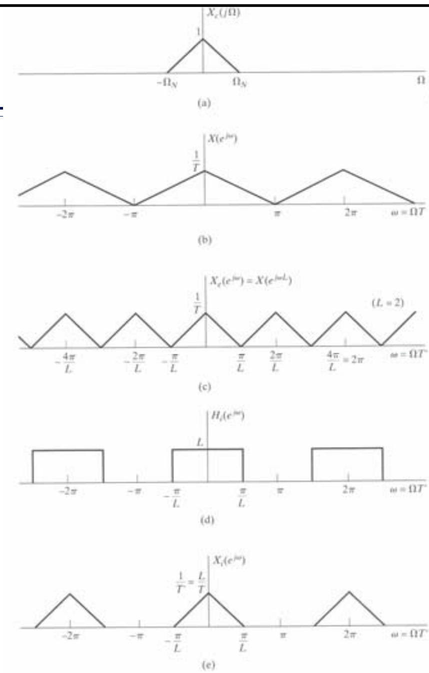


Figure 4.25 Frequency-domain illustration of interpolation.

Summary

- Periodic sampling
- Frequency domain representation
- Reconstruction
- Changing the sampling rate using discrete-time processing

Course at a glance