Digital Signal Processing

http://kom.aau.dk/~zt/cources/DSP/

Solutions of Lecture 3 (MM3)

Exercise 3.1.

a) $$X(z)=\sum_{n=0}^{\infty}(\frac{1}{2})^n z^{-n} = \frac{z}{z-\frac{1}{2}}$$

For convergence of $X(z)$, require that $\sum_{n=0}^{\infty}|\frac{1}{2}z^{-1}|^n < \infty$.

So, ROC is $|z|>\frac{1}{2}$.

b) $$X(z)=-\sum_{n=-\infty}^{-1}(½)^n z^{-n} = \frac{z}{z-\frac{1}{2}}, \quad |z|<\frac{1}{2}.$$

c) $$X(z)=\frac{1}{1-2z}, \quad |z|<\frac{1}{2}.$$

d) $$X(z)=z^{-1}Z(\delta[n])=z^{-1}, \quad |z|>0.$$

e) $$X(z)=z, \quad |z|<\infty.$$

f) $$X(z)=\sum_{n=0}^{9}(\frac{1}{2z})^n = \frac{1-(\frac{1}{2z})^{10}}{1-\frac{1}{2z}}, \quad |z|>0.$$

g) $$X(z)=\sum_{n=-\infty}^{-1}\alpha^{-n}z^{-n}+\sum_{n=0}^{\infty}\alpha^n z^{-n} = \frac{\alpha z}{1-\alpha z}+\frac{z}{z-\alpha}, \quad |\alpha|<|z|<\frac{1}{|\alpha|}.$$

Exercise 3.2. The input to a LTI-system is $u[n]$ and the output $y[n]$ is $y[n]=\left(\frac{1}{2}\right)^{n-1}u[n+1]$.
Find the transfer function, H(z).

Solution:

$$X(z)=\frac{1}{1-z^{-1}}, \quad |z|>1.$$

$$Y(z)=\frac{4z}{1-\frac{1}{2}z^{-1}}, \quad |z|>\frac{1}{2}.$$

$$H(z)=\frac{4z(1-z^{-1})}{1-\frac{1}{2}z^{-1}},$$

Exercise 3.3. Find the Z-transform of the sequence

$$x[n] = \begin{cases} n, & 0 \le n \le \mathrm{N}-1, \\ N, & N \le n. \end{cases}$$

Hint: Express $x[n]$ using $u[n]$.

Solution:

$$x[n] = nu[n] - (n-N)u(n-N)$$

$$X(z) = \frac{z^{-1}}{(1-z^{-1})^2} - \frac{z^{-N-1}}{(1-z^{-1})^2} = \frac{z^{-1}(1-z^{-N})}{(1-z^{-1})^2}$$

Exercise 3.4. Given the Z-transform $X(z) = \dfrac{1}{(1+\frac{1}{2}z^{-1})}, |z| > \frac{1}{2}$. Use power series expansion (long division) to find x[n].

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$

Partial fractions: one pole → inspection, $x[n] = (-\frac{1}{2})^n u[n]$

Long division:

$$\begin{array}{r|llll} & 1 & -\frac{1}{2}z^{-1} & +\frac{1}{4}z^{-2} & + \dots \\ \hline 1+\frac{1}{2}z^{-1} & 1 & & & \\ & 1 & +\frac{1}{2}z^{-1} & & \\ \hline & & -\frac{1}{2}z^{-1} & & \\ & & -\frac{1}{2}z^{-1} & -\frac{1}{4}z^{-2} & \\ \hline & & & +\frac{1}{4}z^{-2} & \\ & & & +\frac{1}{4}z^{-2} & +\frac{1}{8}z^{-3} \\ \hline \end{array}$$

$$\Longrightarrow x[n] = \left(-\frac{1}{2}\right)^n u[n]$$

Exercise 3.5. Given the Z-transform $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{(1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2})}, |z| > \frac{1}{2}$. Try both using partial fraction expansion and power series expansion (long division) to find x[n].

$$X(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$X(z) = \frac{-3}{1+\frac{1}{4}z^{-1}} + \frac{4}{1+\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$

$$x[n] = \left[-3\left(-\frac{1}{4}\right)^n + 4\left(-\frac{1}{2}\right)^n\right]u[n]$$

Longdivision:

$$\begin{array}{r|llll}
 & 1 & +\,(-\frac{3}{4}-\frac{1}{2})z^{-1} & +\,(-\frac{3}{16}+1)z^{-2} & -\ \dots \\
\hline
1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2} & 1 & -\frac{1}{2}z^{-1} & & \\
 & 1 & +\frac{3}{4}z^{-1} & +\frac{1}{8}z^{-2} & \\
\hline
 & & (-\frac{3}{4}-\frac{1}{2})z^{-1} & -\frac{1}{8}z^{-2} & \\
 & & (-\frac{3}{4}-\frac{1}{2})z^{-1} & +\frac{3}{4}(-\frac{3}{4}-\frac{1}{2})z^{-2} & +\frac{1}{8}(-\frac{3}{4}-\frac{1}{2})z^{-3} \\
\hline
 & & & [-\frac{1}{8}+\frac{3}{4}(\frac{3}{4}+\frac{1}{2})]z^{-2} & +\frac{1}{8}(\frac{3}{4}+\frac{1}{2})z^{-3}
\end{array}$$

$$\Longrightarrow x[n] = \left[-3(\left(-\frac{1}{4}\right)^n + \left(-\frac{1}{2}\right)^{n-2}\right]u[n]$$

Exercise 3.6. Given the transfer function $H(z) = \dfrac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$ of a causal LTI system. Is the system stable ? (First find the ROC)

Solution:

$H(z) = \dfrac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})}$, causal and LTI,

Since system is causal, ROC is $|z| > \dfrac{1}{2}$, which includes the unit circle, the system is stable.