

# Digital Signal Processing, Fall 2006

## Lecture 3: The z-transform

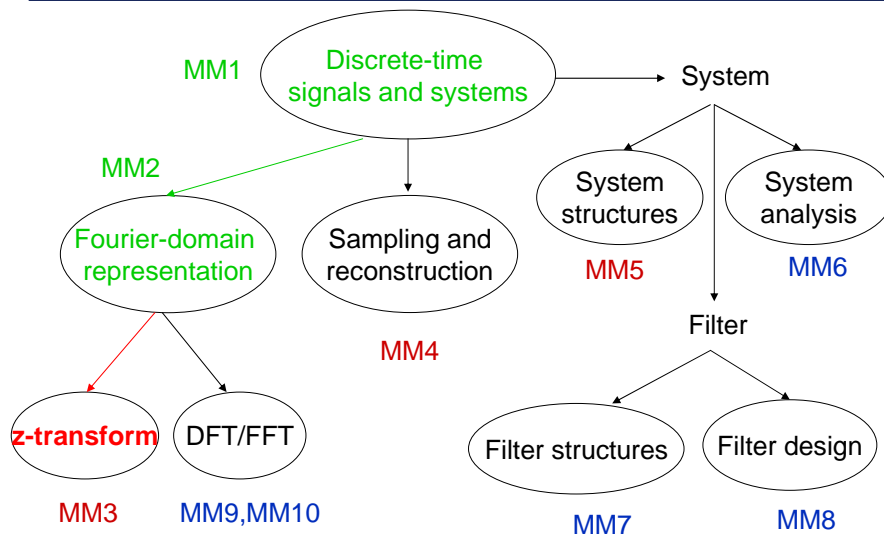
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## Course at a glance



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## Part I: z-transform

- z-transform
- Properties of the ROC
- Inverse z-transform
- Properties of z-transform

## Limitation of Fourier transform

- Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

- Condition for the convergence of the infinite sum

$$|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| |e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- If  $x[n]$  is absolutely summable, its Fourier transform exists (sufficient condition).

- Example

$$x[n] = a^n u[n] \quad |a| < 1: X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$a = 1: X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega + 2\pi k)$$

$$|a| > 1: -$$

## z-transform

- Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad x[n] \xleftrightarrow{Z} X(z)$$

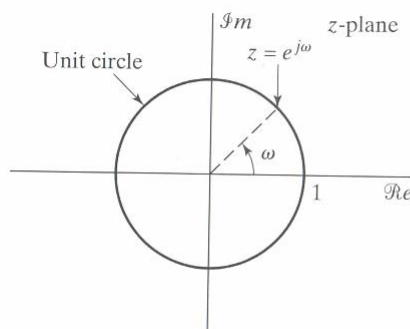
- The complex variable z in polar form  $z = re^{j\omega}$

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

$$|z| = r = 1, \quad X(z) = X(e^{j\omega})$$

## z-plane

- z-transform is a function of a complex variable  $\rightarrow$  using the complex z-plane



Z-transform on unit circle  
 $\leftrightarrow$  Fourier transform

Linear frequency axis in  
 Fourier transform  
 $\rightarrow$  Unit circle in z-transform  
 (periodicity in freq. of  
 Fourier transform)

Figure 3.1 The unit circle in the complex z-plane.

## Region of convergence – ROC

- Fourier transform does not converge for all sequences

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- z-transform does not converge for all sequences or for all values of z.

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} (x[n] r^{-n}) e^{-j\omega n}$$

- ROC – for any given seq., the set of values of z for which the z-transform converges

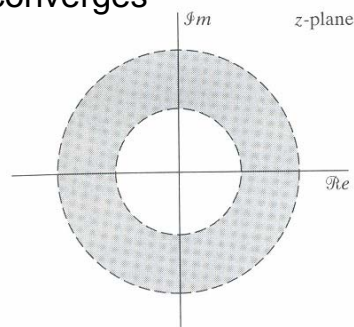
$$\sum_{n=-\infty}^{\infty} |x[n] r^{-n}| < \infty \quad \sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n} < \infty \quad \text{ROC is ring!}$$

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## ROC

- Outer boundary is a circle (may extend to infinity)
- Inner boundary is a circle (may extend to include the origin)
- If ROC includes unit circle, Fourier transform converges



**Figure 3.2** The region of convergence (ROC) as a ring in the z-plane. For specific cases, the inner boundary can extend inward to the origin, and the ROC becomes a disc. For other cases, the outer boundary can extend outward to infinity.

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## Zeros and poles

- The most important and useful z-transforms – rational function:

$$X(z) = \frac{P(z)}{Q(z)}$$

$P(z)$  and  $Q(z)$  are polynomials in  $z$

- Zeros: values of  $z$  for which  $X(z)=0$ .
- Poles : values of  $z$  for which  $X(z)$  is infinite.
- Close relation between poles and ROC

## Right-sided exponential sequence

$$x[n] = a^n u[n]$$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} (az^{-1})^n \end{aligned}$$

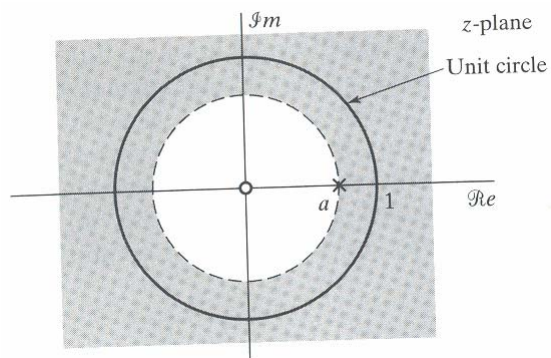
- ROC

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$$

- z-transform

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

$$u[n] \stackrel{z}{\leftrightarrow} \frac{1}{1-z^{-1}}, \quad |z| > 1$$



## Left-sided exponential sequence

$$x[n] = -a^n u[-n-1]$$

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} \\ &= -\sum_{n=-\infty}^{-1} a^n z^{-n} = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n \end{aligned}$$

### ■ ROC

$$\sum_{n=0}^{\infty} |a^{-1} z|^n < \infty$$

### ■ z-transform

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| < |a|$$

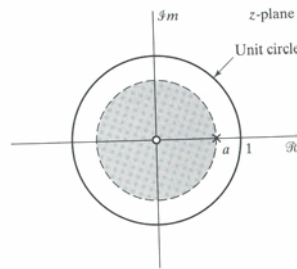


Figure 3.4 Pole-zero plot and region of convergence for Example 3.2.

## Sum of two exponential sequence

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

### ■ ROC

$$\left|\frac{1}{2} z^{-1}\right| < 1 \text{ and } \left|-\frac{1}{3} z^{-1}\right| < 1$$

$$|z| > \frac{1}{2} \text{ and } |z| > \frac{1}{3}$$

$$|z| > \frac{1}{2}$$

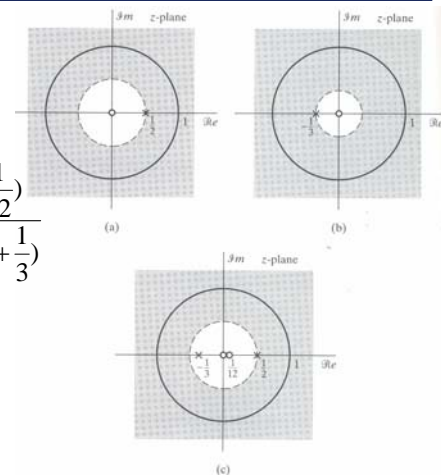


Figure 3.5 Pole-zero plot and region of convergence for the individual terms and the sum of terms in Examples 3.3 and 3.4. (a)  $1/(1 - \frac{1}{2} z^{-1})$ ,  $|z| > \frac{1}{2}$ . (b)  $1/(1 + \frac{1}{3} z^{-1})$ ,  $|z| > \frac{1}{3}$ . (c)  $1/(1 - \frac{1}{2} z^{-1}) + 1/(1 + \frac{1}{3} z^{-1})$ ,  $|z| > \frac{1}{2}$ .

## Sum of two exponential sequence

Another way to calculate:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

$$\left(\frac{1}{2}\right)^n u[n] \stackrel{z}{\leftrightarrow} \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$\left(-\frac{1}{3}\right)^n u[n] \stackrel{z}{\leftrightarrow} \frac{1}{1 + \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \stackrel{z}{\leftrightarrow} \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}} \quad |z| > \frac{1}{2}$$

## Two-sided exponential sequence

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\left(-\frac{1}{3}\right)^n u[n] \stackrel{z}{\leftrightarrow} \frac{1}{1 + \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \stackrel{z}{\leftrightarrow} \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$

$$X(z) = \frac{1}{1 + \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{3}, \quad |z| < \frac{1}{2}$$

$$= \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

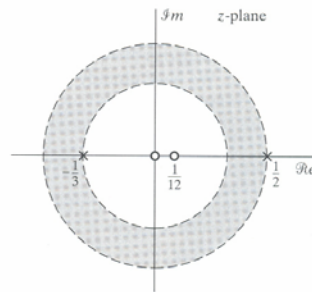


Figure 3.6 Pole-zero plot and region of convergence for Example 3.5.

## Finite-length sequence

$$x[n] = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n \\ &= \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a} \end{aligned}$$

$$\begin{aligned} \text{ROC } \sum_{n=0}^{N-1} |az^{-1}|^n < \infty \\ |a| < \infty \text{ and } z \neq 0 \end{aligned}$$

$$z_k = ae^{j(2\pi k/N)}, \quad k = 0, 1, \dots, N-1$$

pole at  $z = a$

$$z_k = ae^{j(2\pi k/N)}, \quad k = 1, \dots, N-1$$

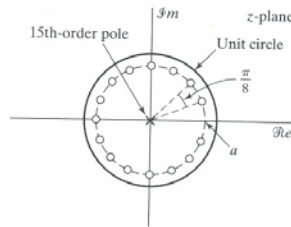


Figure 3.7 Pole-zero plot for Example 3.6 with  $N = 16$  and  $a$  real such that  $0 < a < 1$ . The region of convergence in this example consists of all values of  $z$  except  $z = 0$ .

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All $z$
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$ z  > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$ z  < 1$
4. $\delta[n-m]$	$z^{-m}$	All $z$ except 0 (if $m > 0$ ) or $\infty$ (if $m < 0$ )
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$ z  >  a $
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$ z  <  a $
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z  >  a $
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z  <  a $
9. $[\cos \omega_0 n] u[n]$	$\frac{1 - [\cos \omega_0] z^{-1}}{1 - [2 \cos \omega_0] z^{-1} + z^{-2}}$	$ z  > 1$
10. $[\sin \omega_0 n] u[n]$	$\frac{[\sin \omega_0] z^{-1}}{1 - [2 \cos \omega_0] z^{-1} + z^{-2}}$	$ z  > 1$
11. $[r^n \cos \omega_0 n] u[n]$	$\frac{1 - [r \cos \omega_0] z^{-1}}{1 - [2r \cos \omega_0] z^{-1} + r^2 z^{-2}}$	$ z  > r$
12. $[r^n \sin \omega_0 n] u[n]$	$\frac{[r \sin \omega_0] z^{-1}}{1 - [2r \cos \omega_0] z^{-1} + r^2 z^{-2}}$	$ z  > r$
13. $\begin{cases} a^n, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z  > 0$



## Part II: Properties of the ROC

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- z-transform
- Properties of the ROC
- Inverse z-transform
- Properties of z-transform

PROPERTY 1: The ROC is a ring or disk in the  $z$ -plane centered at the origin; i.e.,  $0 \leq r_R < |z| < r_L \leq \infty$ .

PROPERTY 2: The Fourier transform of  $x[n]$  converges absolutely if and only if the ROC of the  $z$ -transform of  $x[n]$  includes the unit circle.

PROPERTY 3: The ROC cannot contain any poles.

PROPERTY 4: If  $x[n]$  is a *finite-duration sequence*, i.e., a sequence that is zero except in a finite interval  $-\infty < N_1 \leq n \leq N_2 < \infty$ , then the ROC is the entire  $z$ -plane, except possibly  $z = 0$  or  $z = \infty$ .

PROPERTY 5: If  $x[n]$  is a *right-sided sequence*, i.e., a sequence that is zero for  $n < N_1 < \infty$ , the ROC extends outward from the *outermost* (i.e., largest magnitude) finite pole in  $X(z)$  to (and possibly including)  $z = \infty$ .

PROPERTY 6: If  $x[n]$  is a *left-sided sequence*, i.e., a sequence that is zero for  $n > N_2 > -\infty$ , the ROC extends inward from the *innermost* (smallest magnitude) nonzero pole in  $X(z)$  to (and possibly including)  $z = 0$ .

PROPERTY 7: A *two-sided sequence* is an infinite-duration sequence that is neither right sided nor left sided. If  $x[n]$  is a two-sided sequence, the ROC will consist of a ring in the  $z$ -plane, bounded on the interior and exterior by a pole and, consistent with property 3, not containing any poles.

PROPERTY 8: The ROC must be a connected region.

## Properties of the ROC

- The algebraic expression or pole-zero pattern does not completely specify the z-transform of a sequence → the ROC must be specified!
- Stability, causality and the ROC

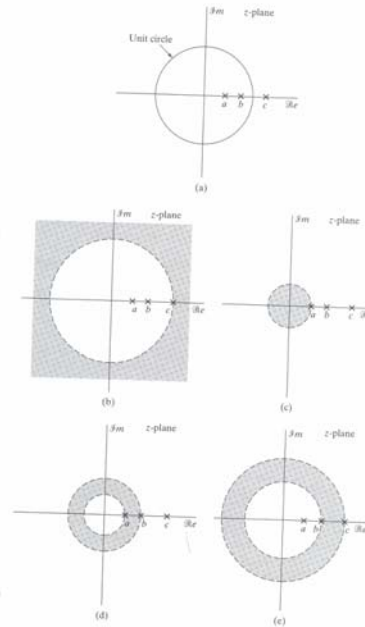


Figure 3.10 Examples of four z-transforms with the same pole-zero locations, illustrating the different possibilities for the region of convergence. Each ROC corresponds to a different sequence: (b) to a right-sided sequence, (c) to a left-sided sequence, (d) to a two-sided sequence, and (e) to a two-sided sequence.

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## Part III: Inverse z-transform

- z-transform
- Properties of the ROC
- Inverse z-transform
- Properties of z-transform

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## Inverse z-transform

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- Needed for system analysis: 1) z-transform, 2) manipulation, 3) inverse z-transform.
- Approaches:
  - Inspection method
  - Partial fraction expansion
  - Power series expansion

## Inspection method

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- By inspection, e.g.

$$X(z) = \left( \frac{1}{1 - \frac{1}{2}z^{-1}} \right), \quad |z| > \frac{1}{2}$$

- Make use of

$$a^n u(n) \stackrel{z}{\leftrightarrow} \left( \frac{1}{1 - az^{-1}} \right), \quad |z| > |a|$$

$$\therefore x[n] = \left( \frac{1}{2} \right)^n u[n]$$

$$\text{if } |z| < \frac{1}{2} \text{?} \quad \text{of course, } x[n] = -\left( \frac{1}{2} \right)^n u[-n-1]$$

## Partial fraction expansion

- For rational function, get the format of a sum of simpler terms, and then use the inspection method.

## Second-order z-transform

$$X(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}, \quad |z| > \frac{1}{2}$$

$$X(z) = \frac{1}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})}$$

$$X(z) = \frac{A_1}{(1 - \frac{1}{4}z^{-1})} + \frac{A_2}{(1 - \frac{1}{2}z^{-1})}$$

$$A_1 = (1 - \frac{1}{2}z^{-1})X(z)|_{z=1/4} = -1$$

$$A_2 = (1 - \frac{1}{4}z^{-1})X(z)|_{z=1/2} = 2$$

$$X(z) = \frac{-1}{(1 - \frac{1}{4}z^{-1})} + \frac{2}{(1 - \frac{1}{2}z^{-1})}$$

$$x[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

$$X(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

$$X(z) = \frac{b_0 \sum_{k=1}^M (1 - c_k z^{-1})}{a_0 \sum_{k=1}^N (1 - d_k z^{-1})}$$

$$X(z) = \sum_{k=1}^N \frac{A_k}{1 - d_k z^{-1}}$$

$$A_k = X(z)(1 - d_k z^{-1})|_{z=d_k}$$

## What about $M \geq N$ ?

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}, \quad |z| > 1 \qquad X(z) = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}$$

$$X(z) = B_0 + \frac{A_1}{(1 - \frac{1}{2}z^{-1})} + \frac{A_2}{(1 - z^{-1})} \qquad \frac{\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1}{z^{-2} + 2z^{-1} + 1}$$

$$B_0 = 2 \quad \text{Found by long division.}$$

$$A_1 = (1 - \frac{1}{2}z^{-1})X(z)|_{z=1/2} = -9$$

$$A_2 = (1 - z^{-1})X(z)|_{z=1} = 8$$

$$X(z) = 2 - \frac{9}{(1 - \frac{1}{2}z^{-1})} + \frac{8}{(1 - z^{-1})} \qquad x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

## Power series expansion

- By long division

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$\frac{1 + az^{-1} + a^2z^{-2} + \dots}{1 - az^{-1}} \Bigg| 1$$

$$\frac{1 - az^{-1}}{az^{-1}}$$

$$az^{-1}$$

$$\frac{az^{-1} - a^2z^{-2}}{a^2z^{-2} \dots}$$

$$a^2z^{-2} \dots$$

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \dots$$

$$x[n] = a^n u[n]$$

## Finite-length sequence

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$$X(z) = z^2 \left(1 - \frac{1}{2}z^{-1}\right) \left(1 + z^{-1}\right) \left(1 - z^{-1}\right)$$

$$X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

## Part IV: Properties of z-transform

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- z-transform
- Properties of the ROC
- Inverse z-transform
- Properties of z-transform

## Linearity

$$x_1[n] \leftrightarrow X_1(z), \text{ ROC} = R_{x_1}$$

$$x_2[n] \leftrightarrow X_2(z), \text{ ROC} = R_{x_2}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

### Linearity

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z), \text{ ROC contains } R_{x_1} \cap R_{x_2} \text{ at least}$$

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}}, |z| > 1$$

$$u[n-1] \leftrightarrow \frac{z^{-1}}{1-z^{-1}}, |z| > 1$$

$$u[n] - u[n-1] = \delta[n] \leftrightarrow \frac{1-z^{-1}}{1-z^{-1}} = 1, \text{ All } z$$

## Time shifting

$$x[n - n_0] \leftrightarrow z^{-n_0} X(z),$$

$$\text{ROC} = R_{x_1} \text{ (except for 0 or } \infty)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

### Example

$$X(z) = \frac{1}{z - \frac{1}{4}}, |z| > \frac{1}{4}$$

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}} = z^{-1} \left( \frac{1}{1 - \frac{1}{4}z^{-1}} \right)$$

$$\left(\frac{1}{4}\right)^n u[n] \leftrightarrow \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$$\left(\frac{1}{4}\right)^{n-1} u[n-1] \leftrightarrow z^{-1} \left( \frac{1}{1 - \frac{1}{4}z^{-1}} \right)$$

## Multiplication by exponential sequence

$$z_0^n x[n] \stackrel{Z}{\leftrightarrow} X(z/z_0), \quad \text{ROC} = |z_0| R_x$$

$$e^{j\omega_0 n} x[n] \stackrel{F}{\leftrightarrow} X(e^{j(\omega-\omega_0)})$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

### Examples

$$u[n] \stackrel{Z}{\leftrightarrow} \frac{1}{1-z^{-1}}, \quad |z| > 1$$

$$a^n u[n] \stackrel{Z}{\leftrightarrow} \frac{1}{1-(z/a)^{-1}} = \frac{1}{1-az^{-1}}, \quad |z| > a$$

$$x[n] = \cos(\omega_0 n) u[n] = \frac{1}{2} (e^{j\omega_0})^n u[n] + \frac{1}{2} (e^{-j\omega_0})^n u[n]$$

$$X(z) = \frac{\frac{1}{2}}{1-e^{j\omega_0} z^{-1}} + \frac{\frac{1}{2}}{1-e^{-j\omega_0} z^{-1}} = \frac{(1-\cos \omega_0 z^{-1})}{1-2\cos \omega_0 z^{-1} + z^{-2}}, \quad |z| > 1$$

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## Differentiation of X(z)

$$nx[n] \stackrel{Z}{\leftrightarrow} -z \frac{dX(z)}{dz}, \quad \text{ROC} = R_x$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

■ Example  $X(z) = \log(1+az^{-1}), \quad |z| > |a|$

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1+az^{-1}}$$

$$nx[n] \stackrel{Z}{\leftrightarrow} -z \frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}}, \quad |z| > |a|$$

$$nx[n] = a(-a)^{n-1} u[n-1]$$

$$x[n] = \frac{a(-a)^{n-1}}{n} u[n-1]$$

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## Conjugation of a complex sequence

$$x^*[n] \stackrel{z}{\leftrightarrow} X^*(z^*), \text{ ROC} = R_x$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

## Time reversal

$$x[-n] \stackrel{z}{\leftrightarrow} X(1/z), \text{ ROC} = 1/R_x$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

### ■ Example

$$x[n] = a^{-n}u[-n]$$

$$X(z) = \frac{1}{1-az}, \quad |z| < |a^{-1}|$$

$$x[n] = a^n u[n]$$

$$X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

## Convolution of sequences

$$x_1[n] * x_2[n] \stackrel{Z}{\leftrightarrow} X_1(z)X_2(z),$$

ROC contains  $R_{x_1} \cap R_{x_2}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

### ■ Example

$$x[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$h[n] = u[n]$$

$$y[n]?$$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

$$H(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

$$Y(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}, \quad |z| > \frac{1}{2}$$

$$y[n] = \frac{1}{1 - \frac{1}{2}} \left( u[n] - \left(\frac{1}{2}\right)^{n+1} u[n] \right) \leftarrow = \frac{1}{1 - \frac{1}{2}} \left( \frac{1}{(1 - z^{-1})} - \frac{\frac{1}{2}}{(1 - \frac{1}{2}z^{-1})} \right)$$

## Initial-value theorem

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\lim_{z \rightarrow \infty} X(z) = \lim_{z \rightarrow \infty} \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \lim_{z \rightarrow \infty} z^{-n}$$

$$= x[0]$$

**TABLE 3.2** SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	$R_x$
		$x_1[n]$	$X_1(z)$	$R_{x_1}$
		$x_2[n]$	$X_2(z)$	$R_{x_2}$
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	$R_x$ , except for the possible addition or deletion of the origin or $\infty$
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$ z_0  R_x$
4	3.4.4	$nx[n]$	$-z \frac{dX(z)}{dz}$	$R_x$ , except for the possible addition or deletion of the origin or $\infty$
5	3.4.5	$x^*[n]$	$X^*(z^*)$	$R_x$
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains $R_x$
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains $R_x$
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
10	3.4.8	Initial-value theorem: $x[n] = 0, \quad n < 0 \quad \lim_{z \rightarrow \infty} X(z) = x[0]$		

## Summary

- z-transform
- Properties of the ROC
- Inverse z-transform
- Properties of z-transform

# Course at a glance

