Digital Signal Processing, Fall 2006

- E-Study -

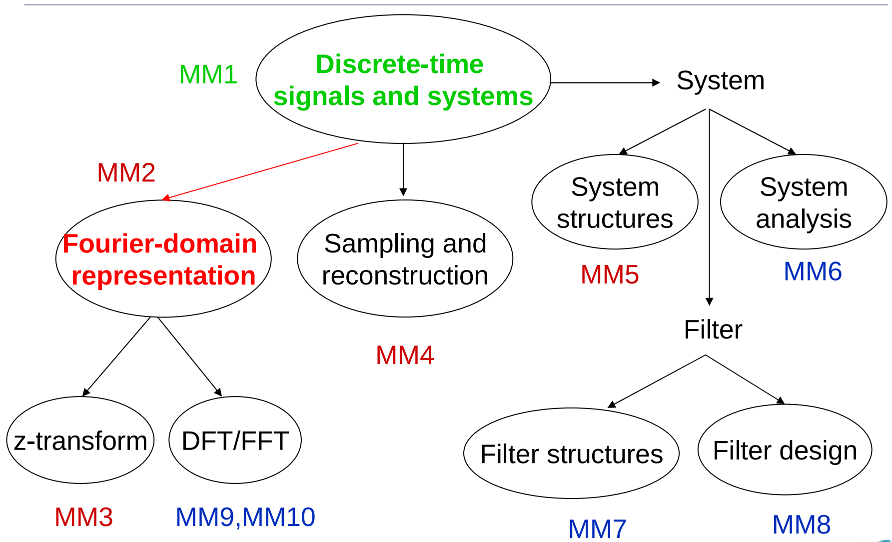
Lecture 2: Fourier transforms and frequency response

Zheng-Hua Tan

Department of Electronic Systems
Aalborg University, Denmark
zt@kom.aau.dk

AALBORG UNIVERSITY

Course at a glance



AALBORG UNIVERSITY

Part I: System impulse response

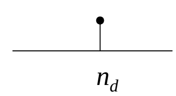
- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response

FIR systems – reflected in the *h*[*n*]

- Ideal delay

$$y[n] = x[n - n_d], \quad -\infty < n < \infty$$

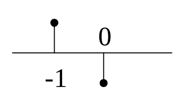
$$\Rightarrow h[n] = \delta[n - n_d], \quad n_d \text{ a positive integer.}$$



- Forward difference

$$y[n] = x[n+1] - x[n]$$

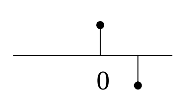
$$\Rightarrow h[n] = \delta[n+1] - \delta[n]$$



- Backward difference

$$y[n] = x[n] - x[n-1]$$

$$\Rightarrow h[n] = \delta[n] - \delta[n-1]$$



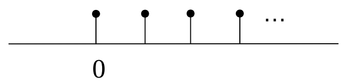
- Finite-duration impulse response (FIR) system
  - The impulse response has only a finite number of nonzero samples.

IIR systems – reflected in the $h[n]$

- Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$\Rightarrow \quad h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n]$$



- Infinite-duration impulse response (IIR) system
  - The impulse response is infinitive in duration.
- Stability $S = \sum_{n=-\infty}^{\infty} |h[n]| \overset{?}{<} \infty$
  - FIR systems always are stable, if each of h[n] values is finite in magnitude.
  - IIR systems can be stable, e.g. $h[n] = a^n u[n]$ with $|a| < 1$

$$S = \sum_{0}^{\infty} |a|^n = 1/(1-|a|) < \infty$$

Cascading systems

- Causality $h[n] \overset{?}{=} 0, \quad n < 0$
  - Ideal delay $h[n] = \delta[n - n_d]$

$x[n] \longrightarrow$ Forward difference $h[n] = \delta[n+1] - \delta[n]$ $\longrightarrow$ One-sameple delay $h[n] = \delta[n-1]$ $\longrightarrow y[n]$

⇓

$x[n] \longrightarrow$ Backward difference $h[n] = \delta[n] - \delta[n-1]$ $\longrightarrow y[n]$

  - Any noncausal FIR system can be made cause by cascading it with a sufficiently long delay!

Cascading systems

- Accumulator + Backward difference system

$x[n] \longrightarrow$ Accumulator system $h[n] = u[n]$ $\longrightarrow$ Backward difference $h[n] = \delta[n] - \delta[n-1]$ $\longrightarrow x[n]$

⇓

$x[n] \longrightarrow x[n]$, $h[n] = \delta[n]$

Inverse system:

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n]$$

Part II: LCCD equations

- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response

LCCD equations

- An important class of LTI systems: input and output satisfy an Nth-order LCCD equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

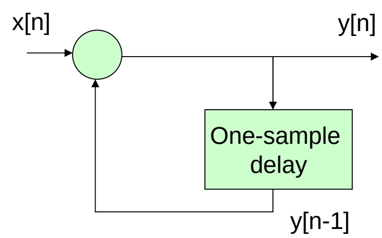
- Difference equation representation of the accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] = x[n] + \sum_{k=-\infty}^{n-1} x[k] = x[n] + y[n-1]$$

$$y[n] - y[n-1] = x[n]$$



Recursive representation

Part III: Fourier transforms

- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response
  - Frequency-domain representation of discrete-time signals and systems
  - Symmetry properties of the Fourier transform
  - Fourier transform theorems

Signal representations

- A sum of scaled, delayed impulse

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad \Longrightarrow \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Sinusoidal and complex exponential sequences
  - Sinusoidal input → sinusoidal response with the same frequency and with amplitude and phase determined by the system $x[n] = A\cos(\omega_0 n + \phi)$
  - Complex exponential sequences are eigenfunctions of LTI systems. $x[n] = e^{j\omega n}$

  → signal representation based on sinusoids or complex exponentials

Eigenfunctions

Complex exponentials as input to system h[n]

$$x[n] = e^{j\omega n}, \quad -\infty < n < \infty$$

$$y[n] = T\{e^{j\omega n}\} = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)}$$

$$= (\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k})\, e^{j\omega n}$$

$$\mathcal{A}f = \lambda f$$

A – linear operator

Define $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$

Then $y[n] = H(e^{j\omega})e^{j\omega n}$

Eigenvalue

Eigenfunction

Eigenvalue – called frequency response

- Frequency response is generally complex

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$
$$= |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$ describes changes in magnitude and phase

- Frequency response of the ideal delay system

$$y[n] = x[n-n_d] \rightarrow h[n] = \delta[n-n_d]$$

  - Method 1: using the eigenfunction

$$y[n] = T\{e^{j\omega n}\} = e^{j\omega(n-n_d)} = e^{-j\omega n_d} e^{j\omega n}$$
$$H(e^{j\omega}) = e^{-j\omega n_d}$$

  - Method 2: using the impulse response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n-n_d] e^{-j\omega n} = e^{-j\omega n_d}$$

$$H_R(e^{j\omega}) = \cos(\omega n_d),$$
$$H_I(e^{j\omega}) = -\sin(\omega n_d).$$
$$|H(e^{j\omega})| = 1,$$
$$\angle H(e^{j\omega}) = -\omega n_d.$$

Frequency response

- The frequency response of discrete-time LTI systems is always a periodic function of the frequency variable w with period 2π.

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} e^{-j2\pi n} = H(e^{j\omega})$$

  - Only specify over the interval $-\pi < \omega \le \pi$
  - The 'low frequencies' are close to 0
  - The 'high frequencies' are close to $\pm\pi$

Ideal frequency-selective filters

- For which the frequency response is unity over a certain range of frequencies, and is zero at the remaining frequencies.
  - Ideal low-pass filter: passes only low and rejects high

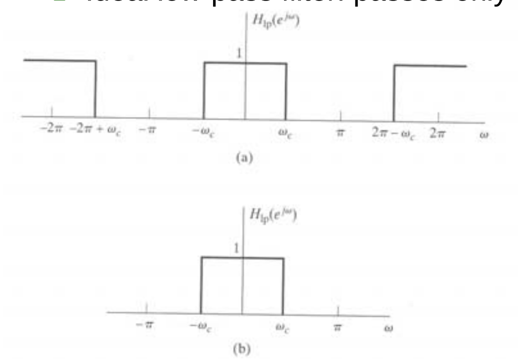


Figure 2.17 Ideal lowpass filter showing (a) periodicity of the frequency response and (b) one period of the periodic frequency response.

AALBORG UNIVERSITY

Ideal frequency-selective filters

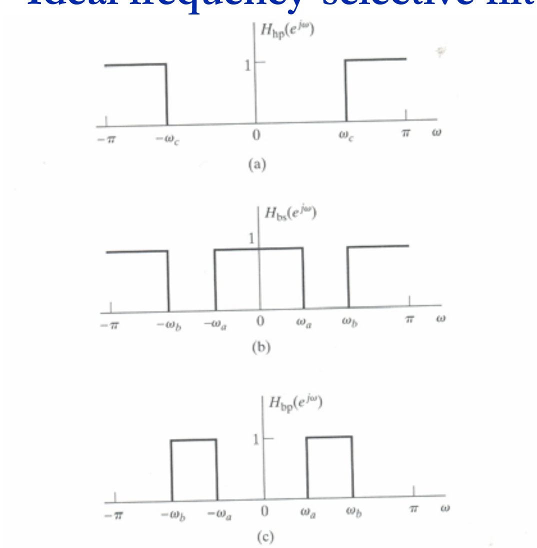


Figure 2.18 Ideal frequency-selective filters. (a) Highpass filter. (b) Bandstop filter. (c) Bandpass filter. In each case, the frequency response is periodic with period 2π. Only one period is shown.

AALBORG UNIVERSITY

Sinusoidal response of LTI systems

- Sinusoidal input → sinusoidal response with the same frequency and with amplitude and phase determined by the system

$$x[n] = A\cos(\omega_0 n + \phi)$$

$$= \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

$$y[n] = \frac{A}{2}e^{j\phi}H(e^{j\omega_0})e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}H(e^{-j\omega_0})e^{-j\omega_0 n}$$

if $h[n]$ is real, then

$$H(e^{-j\omega_0}) = H^*(e^{j\omega_0}) = |H(e^{j\omega_0})|e^{-\angle H(e^{j\omega_0})} = |H(e^{j\omega_0})|e^{-\theta}$$

then $y[n] = A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta)$

Signal representation

- More than sinusoids, a broad class of signals can be represented as a linear combination of complex exponentials:

$$x[n] = \sum_k \alpha_k e^{j\omega_k n}$$

$$\therefore \quad y[n] = \sum_k \alpha_k H(e^{j\omega_k})e^{j\omega_k n}$$

- If x[n] can be represented as a superposition of complex exponentials, output y[n] can be computed by using the frequency response, which is similar to the function of impulse response.

Frequency-domain representation of x[n]

- By Fourier transforms → Fourier representation:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Inverse Fourier transform
w is continuous-time variable

where $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$

Fourier transform
n is discrete-time variable

represent *x[n]* as

a superposition of infinitesimally small complex sinusoids

$$\frac{1}{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- In general, Fourier transform is complex

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$
$$= |X(e^{j\omega})|\, e^{j\angle X(e^{j\omega})}$$

Fourier spectrum
Spectrum
Magnitude spectrum
Amplitude spectrum
Phase spectrum

Frequency and impulse responses

- Are a Fourier transform pair

Recall $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$

$\therefore$ $h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$

- Fourier transform is periodic with period 2π

$$X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$$

Sufficient condition for Fourier transform

- Condition for the convergence of the infinite sum

$$
\begin{aligned}
|X(e^{j\omega})| &= |\sum_{-\infty}^{\infty} x[n] e^{-j\omega n}| \\
&\le \sum_{-\infty}^{\infty} |x[n]| |e^{-j\omega n}| \\
&\le \sum_{-\infty}^{\infty} |x[n]| < \infty
\end{aligned}
$$

- X[n] is absolutely summable, then its Fourier transform exists (sufficient condition).

Example: ideal lowpass filter

- Frequency response $H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$

$$h_{lp}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

  - Noncausal.
  - Not absolutely summable.

$$H_{lp}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

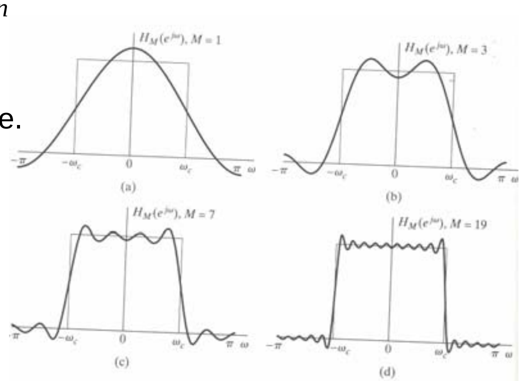


Figure 2.21 Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.

Example: Fourier transform of a constant

- Constant sequence

$$x[n] = 1$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

where $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$

- Its Fourier transform is defined as the periodic impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r)$$

Part III: Fourier transforms

- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response
  - Frequency-domain representation of discrete-time signals and systems
  - Symmetry properties of the Fourier transform
  - Fourier transform theorems

Symmetry properties of Fourier transform

- Conjugate-symmetric sequence $x_e[n] = x_e^*[-n]$
- Conjugate-antisymmetric sequence $x_o[n] = -x_o^*[-n]$
- Any $x[n]$, $x[n] = \frac{1}{2}(x[n] + x^*[-n]) + \frac{1}{2}(x[n] - x^*[-n])$

define $x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n]$

$x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n]$

$\therefore \quad x[n] = x_e[n] + x_o[n]$

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$$

where

$$X_e(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) + X^*(e^{-j\omega})) = X_e^*(e^{-j\omega})$$

$$X_o(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) - X^*(e^{-j\omega})) = -X_o^*(e^{-j\omega})$$

TABLE 2.1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\|X(e^{j\omega})\| = \|X(e^{-j\omega})\|$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Part III: Fourier transforms

- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response
  - Frequency-domain representation of discrete-time signals and systems
  - Symmetry properties of the Fourier transform
  - Fourier transform theorems

Linearity of the Fourier Transform

$$x_1[n] \overset{F}{\leftrightarrow} X_1(e^{j\omega})$$

$$x_2[n] \overset{F}{\leftrightarrow} X_2(e^{j\omega})$$

$$ax_1[n] + bx_2[n] \overset{F}{\leftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Time shifting and frequency shifting

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$x[n-n_d] \overset{F}{\leftrightarrow} e^{-j\omega n_d} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \overset{F}{\leftrightarrow} X(e^{j(\omega-\omega_0)})$$

Time reversal

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$x[-n] \overset{F}{\leftrightarrow} X(e^{-j\omega})$$

if $x[n]$ is real.

$$x[-n] \overset{F}{\leftrightarrow} X^{*}(e^{j\omega})$$

Differentiation in frequency

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$nx[n] \overset{F}{\leftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Parseval's theorem

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

The convolution theorem

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$h[n] \overset{F}{\leftrightarrow} H(e^{j\omega})$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

The modulation or Windowing theorem

$$x[n] \overset{F}{\leftrightarrow} X(e^{j\omega})$$

$$w[n] \overset{F}{\leftrightarrow} W(e^{j\omega})$$

$$y[n] = x[n]w[n]$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

TABLE 2.3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n - n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\displaystyle\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\dfrac{1}{1 - e^{-j\omega}} + \displaystyle\sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a\rvert < 1)$	$\dfrac{1}{(1 - ae^{-j\omega})^2}$
7. $\dfrac{r^n \sin\omega_p(n+1)}{\sin\omega_p}u[n] \quad (\lvert r\rvert < 1)$	$\dfrac{1}{1 - 2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\dfrac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\displaystyle\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\displaystyle\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

Example

- Determining a Fourier transform using Tables 2.2 and 2.3

$$x[n] = a^n u[n-5]$$

$$x_1[n] = a^n u[n] \overset{F}{\leftrightarrow} X_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

$$x_2[n] = x_1[n-5] \qquad (\text{i.e.} = a^{n-5}u[n-5])$$

$$X_2(e^{j\omega}) = e^{-j5\omega} X_1(e^{j\omega}) = \frac{e^{-j5\omega}}{1-ae^{-j\omega}}$$

$$x[n] = a^5 x_2[n] \qquad (\text{i.e.} = a^n u[n-5])$$

$$X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1-ae^{-j\omega}}$$

Summary

- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response
  - Frequency-domain representation of discrete-time signals and systems
  - Symmetry properties of the Fourier transform
  - Fourier transform theorems

Course at a glance